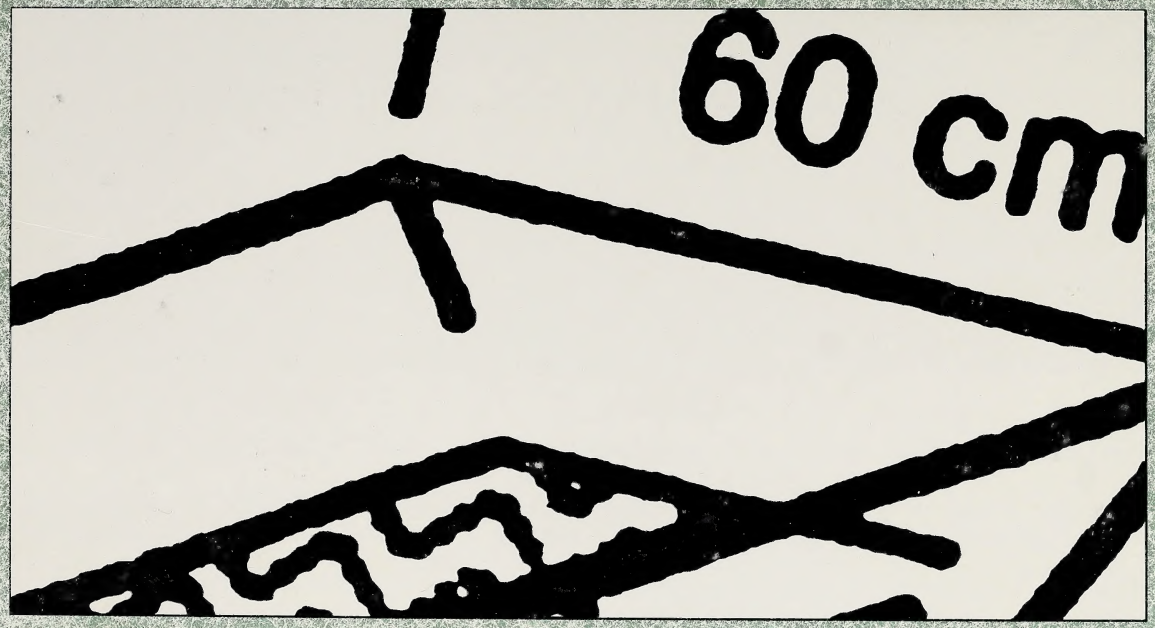
MATHEMATICS 3



Distance Learning

UNIT 3: PRACTICAL APPLICATION OF DERIVATIVES

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Welcome

Distance Learning

You have chosen an alternate form of learning that allows you to work at your own pace. You will be responsible for your own schedule, for disciplining yourself to study the units thoroughly, and for completing your units regularly. We wish you much success and enjoyment in your studies.

CANADIANA

AUG 26 1992

Mathematics 31 Student Module Unit 3 Practical Application to Derivatives Alberta Distance Learning Centre ISBN No. 0-7741-0245-4

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General Information

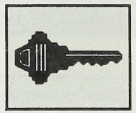
This information explains the basic layout of each booklet.

- **What You Already Know** and **Review** are to help you look back at what you have previously studied. The questions are to jog your memory and to prepare you for the learning that is going to happen in this unit.
- As you begin each **Topic**, spend a little time looking over the components. Doing this will give you a preview of what will be covered in the topic and will set your mind in the direction of learning.
- **Exploring the Topic** includes the objectives, concept development, and activities for each objective. Use your own papers to arrive at the answers in the activities.
- **Extra Help** reviews the topic. If you had any difficulty with **Exploring the Topic**, you may find this part helpful.
- **Extensions** gives you the opportunity to take the topic one step further.
- To summarize what you have learned, and to find instructions on doing the unit assignment, turn to the **Unit Summary** at the end of the unit.
- The **Appendices** include the solutions to **Activities (Appendix A)** and any other charts, tables, etc. which may be referred to in the topics **(Appendix B, etc.)**.

Visual Cues

Visual cues are pictures that are used to identify important areas of the material. They are found throughout the booklet.

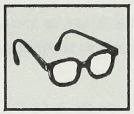
An explanation of what they mean is written beside each visual cue.



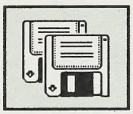
Key Idea
- flagging important ideas

Audiotape
- learning by listening to an audiotape

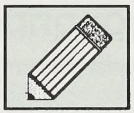
What You Already Know
- reviewing what you already know



Another View
- exploring different perspectives



Computer Software
- learning by using computer software



Review
- studying previous concepts

Solutions
- correcting the activities

Videotape
- learning by viewing a videotape

Introduction
- introducing the unit

Extra Help
- providing additional study

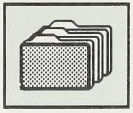
Print Pathway
- choosing a print alternative

What Lies Ahead
- previewing the unit

Extensions
- going on with the topic

Calculator
- using your calculator

Exploring the Topic
- actively learning new concepts



What You Have Learned
- summarizing what you have learned

Mathematics 31

Course Overview

Mathematics 31 contains 9 units. Beside each unit is a percentage that indicates what the unit is worth in relation to the rest of the course. The units and their percentages are listed below. You will be studying the unit that is shaded.

Unit 1 Introduction to Differential Calculus	10%
Unit 2 Differentiation of Algebraic Expressions and Graphing	10%
Unit 3 Practical Application of Derivatives	20%
Unit 4 Integration	10%
Unit 5 Geometric Vectors and Their Application	10%
Unit 6 Algebraic Vectors and Their Application	10%
Unit 7 Inner Product	10%
Unit 8 Systems of Linear Equations	10%
Unit 9 Matrices and Linear Transformations	10%
	100%

Unit Assessment

After completing the unit you will be given a mark based totally on a unit assignment. This assignment will be found in the Assignment Booklet.

Unit Assignment - 100%

If you are working on a CML terminal, your teacher will determine what this assessment will be. It may be

Unit Assignment - 50%
Supervised Unit Test - 50%

Introduction to Practical Application of Derivatives

This unit covers topics dealing with practical application of derivatives. Each topic contains explanations, examples, and activities to assist you in understanding practical application of derivatives. If you find you are having difficulty with the explanations and the way the material is presented, there is a section called **Extra Help**. If you would like to extend your knowledge of the topic, there is a section called **Extensions**.

You can evaluate your understanding of each topic by working through the activities. Answers are found in the solutions in the **Appendix**. In several cases there is more than one way to do the question.

Unit 3 Practical Application of Derivatives

Contents at a Glance

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	• Introduction • Extra Help • What Lies Ahead • Extensions • Exploring Topic 1	
35%	**Topic 2:** Implicit Differentiation and Motion Problems	29
	• Introduction • Extra Help • What Lies Ahead • Extensions • Exploring Topic 2	
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Practical Application of Derivatives

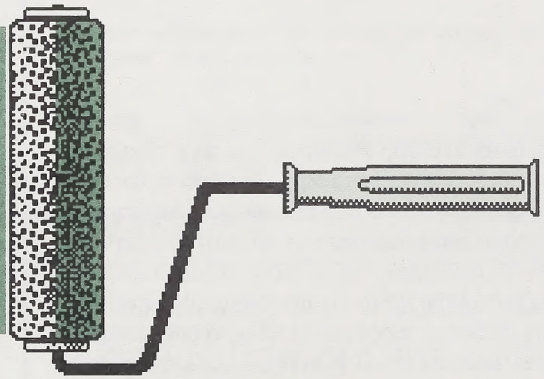
Calculus is very useful in physical sciences, engineering, business, and biological and social sciences. How do you build a container of maximum volume? How can a businessperson maximize profit? How do you determine a maximum bacteria population? How do you determine the velocity and acceleration of a moving object or the rate of change of property or quantity. The practical applications of calculus are immense. You will find many different kinds of applications in this unit.

Unit 3 Practical Application of Derivatives

Topic 1 Maxima and Minima

Topic 2 Implicit Differentiation and Motion Problems

Topic 3 Rates of Change



What You Already Know

Refresh your memory!

Remember what you have learned:

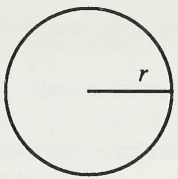
- formulas for perimeters and areas of plane figures including
  - circles
  - triangles
  - squares
  - rectangles
  - parallelograms
  - trapezoids
- formulas for surface areas and volumes of three-dimensional figures including
  - spheres
  - cubes
  - rectangular solids
  - right circular cylinders
  - right circular cones
- properties of similar figures, in particular, similar triangles
- the Pythagorean theorem

Review

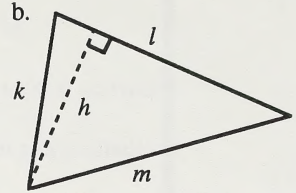
Try the following review questions.

1. Write a formula using some or all of the given variables that could be used to calculate the **area** of each of the following figures.

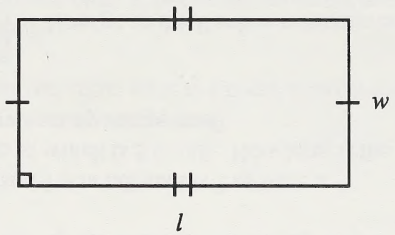
a.



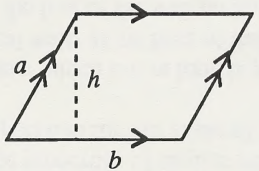
b.



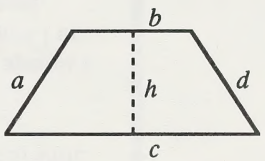
c.



d.



e.



2. Write a formula for the circumference of the circle shown in question 1 a.

3. Define the perimeter of a polygon.

4. In a 30° - 60° - 90° triangle the sides are in the ratio of $1:\sqrt{3}:2$. Use this to show that the area of an equilateral triangle can be found using the formula $A = \frac{\sqrt{3}}{4}s^2$, where s is the length of a side of the triangle.

5. Write formulas for the volume and surface area of a sphere. Find $\frac{dV}{dr}$, where V is the volume of the sphere and r is the radius. What do you notice? Can you explain this result?

6. Write formulas for the surface area and volume of the following:

   a. a closed right circular cylinder
   b. an open-topped cylindrical container

7. Water is being poured into a container shaped like a right circular cone. At any instant the radius of the surface of the water is r units and the depth of the water in the container is h units. Write a formula for the volume of water in the container.

8. If the container in question 7 had a top diameter of 12 cm and a total depth of 8 cm, use the properties of similar triangles to write an expression for r in terms of h.

9. A ladder which is 6 m long is placed against a vertical wall. If the base of the ladder is 1 m from the foot of the wall on level ground, how high up the wall does the ladder reach?

10. A person 1.8 m tall stands 2 m from a lamppost which is 5 m tall. How long is the shadow cast by this person?

Now go to the **Review** solutions in the **Appendix**.

Topic 1 Maxima and Minima

Introduction

Business people want to know how they can get the most profit. Engineers want to know how to run an engine most efficiently. Apartment management firms want to know what rent should be charged if they want to realize the most profit. Calculus helps you answer these types of questions. In this topic you will learn to identify maximum or minimum values and you should be able to solve problems involving maxima or minima.

What Lies Ahead

Throughout the topic you will learn to

1. determine the turning point and second derivative
2. determine the maxima and minima in geometrical problems
3. determine other applications involving maxima and minima

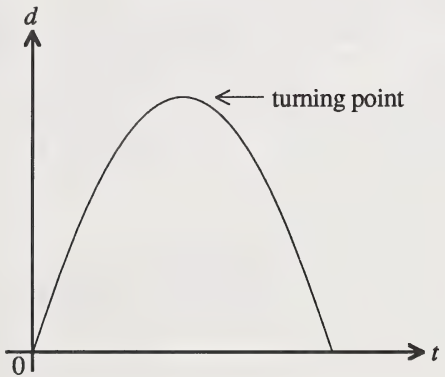
Now that you know what to expect, turn the page to begin your study of maxima and minima.

Exploring Topic 1

Activity 1

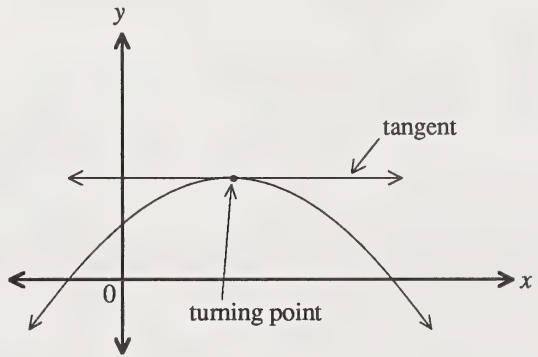
Determine the turning point and second derivative.

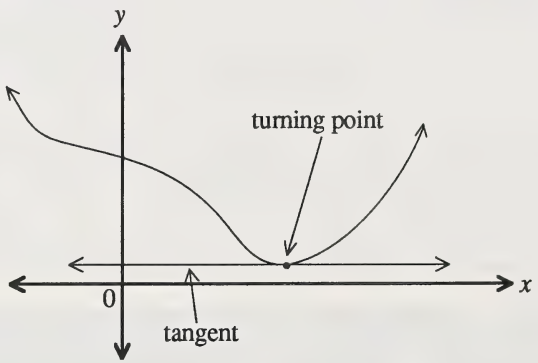
When a ball is thrown upward, it moves higher and higher until it reaches its highest point. Then it starts falling until it hits the ground. The following diagram shows the relation between displacement (d) and time (t).



When the ball is at its highest point, the displacement is a maximum value and the velocity of the ball is zero. This point is called a turning point of the curve. A turning point indicates a maximum or minimum value. You will need to study turning points further.

What is the slope of a horizontal line? Remember that the slope of a horizontal line is zero. If the slope of a line tangent to a curve is zero, then the tangent line must be a horizontal line. When this happens, the curve must have a turning point.





In the previous diagrams you can see that the turning point is a **local maximum** or **local minimum** value of the function (y). It is called **local** because there may be another turning point which represents a greater or smaller value of the function.

In **Mathematics 31, Unit 1**, the first derivative of a function is defined as the slope of the tangent to the curve. According to this definition, $D_x y$ is zero at the turning point on the curve. In other words, you can determine where the turning point is by letting $D_x y$ be zero. You should be aware that a curve can have more than one turning point. Look at the following example.

Example 1

Find the turning points of the curve $y = x^3 - 2x^2 + x - 5$.

Solution:

$$D_x y = 3x^2 - 4x + 1$$

Let $3x^2 - 4x + 1 = 0$

$$(x-1)(3x-1) = 0$$

$$x - 1 = 0, \quad 3x - 1 = 0$$

$$x = 1 \qquad x = \frac{1}{3}$$

When $x = 1$, substitute in the original function.

$$y = 1^3 - 2(1)^2 + 1 - 5$$
$$= -5$$

Therefore, $(1, -5)$ is a turning point.

When $x = \frac{1}{3}$, substitute in the original function.

$$y = \left(\frac{1}{3}\right)^3 - 2\left(\frac{1}{3}\right)^2 + \left(\frac{1}{3}\right) - 5$$
$$= \frac{1}{27} - \frac{2}{9} + \frac{1}{3} - 5$$
$$= -\frac{131}{27}$$
$$= -4\frac{23}{27}$$

Therefore, $\left(\frac{1}{3}, -4\frac{23}{27}\right)$ is another turning point.

The problem is that the first derivative does not distinguish between maxima and minima. It merely indicates which points are turning points. You might be able to distinguish between maxima and minima by plotting other points near the graph of the function, but this method is awkward and may be unreliable should the function have asymptotes. Try this awkward method first. If you want to test the point $(1, -5)$ from the preceding example, you can examine the value of y near this point.

$$\text{At } x = 0.9, \ y = (0.9)^3 - 2(0.9)^2 + 0.9 - 5$$
$$= 0.729 - 1.62 + 0.9 - 5$$
$$= -4.991$$

At $x = 1$, $y = -5$.

At $x = 1.1$, $y = (1.1)^3 - 2(1.1)^2 + 1.1 - 5$

$= 1.331 - 2.42 + 1.1 - 5$

$= -4.989$

The y-value at $x = 1$ is less than its value at all neighbouring points. Therefore, $(1, -5)$ is a minimum point. If the y-value is greater than its value at all neighbouring points, the point is a maximum point. $\left(\frac{1}{3}, -4\frac{23}{27}\right)$ is a maximum point.

Is there a better method? In **Mathematics 31, Unit 2**, you encountered the second derivative. You can easily put it to work to tell you just what you need to know.

In **Mathematics 31, Unit 2**, you learned the following:

- **If the value of the second derivative at a turning point is negative, then that point is a maximum.**
- **If the value of the second derivative at a turning point is positive, then that point is a minimum.**

Test the two turning points in Example 1 by using the second derivative.

$D_x y = 3x^2 - 4x + 1$

$D_x^2 y = 6x - 4$

At $(1, -5)$, $D_x^2 y = 6(1) - 4$

$= 6 - 4$

$= 2$ (positive)

Therefore, $(1, -5)$ is a minimum point.

At $\left(\frac{1}{3}, -4\frac{23}{27}\right)$, $D_x^2 y = 6\left(\frac{1}{3}\right) - 4$

$= 2 - 4$

$= -2$ (negative)

Therefore, $\left(\frac{1}{3}, -4\frac{23}{27}\right)$ is a maximum point.

Look at the next example.

At $x = \frac{1}{4}$,

$y = \left(\frac{1}{4}\right)^3 - 2\left(\frac{1}{4}\right)^2 + \frac{1}{4} - 5$

$\doteq -4.859\,38$

At $x = \frac{1}{3}$,

$y = -4\frac{23}{27}$

$\doteq -4.851\,85$

At $x = \frac{2}{3}$,

$y = \left(\frac{2}{3}\right)^3 - 2\left(\frac{2}{3}\right)^2 + \frac{2}{3} - 5$

$\doteq 0.296\,30 - 0.888\,89 + 0.666\,67 - 5$

$\doteq -4.925\,92$

Example 2

Find the maximum and/or minimum values of $y = 3 - 5x + 4x^2 - x^3$.

Solution:

$$D_x y = -5 + 8x - 3x^2$$

When y is a maximum or minimum, $D_x y = 0$.

$$-5 + 8x - 3x^2 = 0$$
$$3x^2 - 8x + 5 = 0$$
$$(x-1)(3x-5) = 0$$

$$x - 1 = 0, \quad 3x - 5 = 0$$
$$x = 1 \qquad x = \frac{5}{3}$$

At $x = 1$, $y = 3 - 5(1) + 4(1)^2 - 1^3$
$= 1$

Therefore, (1, 1) is a turning point.

At $x = \frac{5}{3}$, $y = 3 - 5\left(\frac{5}{3}\right) + 4\left(\frac{5}{3}\right)^2 - \left(\frac{5}{3}\right)^3$

$$= 3 - \frac{25}{3} + \frac{100}{9} - \frac{125}{27}$$
$$= \frac{81 - 225 + 300 - 125}{27}$$
$$= \frac{31}{27}$$
$$= 1\frac{4}{27}$$

Therefore, $\left(\frac{5}{3}, 1\frac{4}{27}\right)$ is another turning point.

$$D_x^2 y = 8 - 6x$$

At $x = 1$, $D_x^2 y = 8 - 6$
$= 2$

$D_x^2 y$ is positive; therefore, (1, 1) is a minimum point.

The minimum value is 1.

At $x = \frac{5}{3}$, $D_x^2 y = 8 - 6\left(\frac{5}{3}\right)$
$= -2$

$D_x^2 y$ is negative; therefore, $\left(\frac{5}{3}, 1\frac{4}{27}\right)$ is a maximum point.

The maximum value is $1\frac{4}{27}$.

Sometimes you may not be able to find a turning point.

Look at the following example.

Example 3

Find the maximum or minimum values of $y = x\left(5x^2 - 2x + 1\right)$.

Solution:

$$\begin{aligned} D_x y &= x\, D_x\left(5x^2 - 2x + 1\right) + \left(5x^2 - 2x + 1\right) D_x(x) \\ &= x(10x - 2) + \left(5x^2 - 2x + 1\right)(1) \\ &= 10x^2 - 2x + 5x^2 - 2x + 1 \\ &= 15x^2 - 4x + 1 \end{aligned}$$

Let $15x^2 - 4x + 1 = 0$.

$$x = \frac{4 \pm \sqrt{(-4)^2 - 4(15)(1)}}{(2)(15)}$$

$$= \frac{4 \pm \sqrt{-44}}{30}$$

Since these are not real roots, there is no maximum or minimum for real values of x.

Now it is your turn. Do the following questions.

Do at least the odd-numbered questions.

1. Determine the maximum and/or minimum values of the function $y = 3x^2 - 2x + 9$.
2. Determine the maximum and/or minimum values of the function $y = x^3 - 48x + 7$.
3. Determine the maximum and/or minimum values of the function $y = -\frac{1}{4}(x - 2)^2$.
4. Determine the maximum and/or minimum values of the function $y = 2x\left(x^2 - 2x + 1\right)$.
5. Determine the maximum and/or minimum values of the function $y = x^3 - 2x^2 - 4x - 8$.
6. Determine the maximum and/or minimum values of the function $y = 3x\left(x^2 - 2x + 8\right)$.

For solutions to **Activity 1,** turn to the **Appendix, Topic 1.**

Activity 2

Determine the maxima and minima in geometrical problems.

If a farmer wants to build some rectangular cattle pens of 70 m^2 each, the farmer naturally wishes to use the least length of fencing possible. How can the farmer determine the dimensions of the pens? This is the kind of geometrical problem you will find in this section.

Look at an easier example first.

Example 4

Find the dimensions of a rectangle if the perimeter is 28 cm and the area is a maximum. What is the maximum area?

Solution:

Let the length of the rectangle be x. Therefore, the width of the rectangle must be $14 - x$.

$$\begin{aligned} A \text{ (area)} &= l \times w \\ &= x(14 - x) \\ &= 14x - x^2 \end{aligned}$$

$$\begin{aligned} D_x A &= 14 - 2x = 0 \\ 2x &= 14 \\ x &= 7 \end{aligned}$$

$$\begin{aligned} 14 - x &= 14 - 7 \\ &= 7 \end{aligned}$$

Therefore, the dimensions of the rectangle are 7 cm $\times$ 7 cm. The maximum area is 49 cm^2.

The next example involves geometrical measurement in three dimensions.

$$28 = 2l + 2w$$
$$w = 14 - l$$

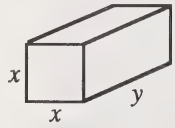
The coefficient of the x^2 term is negative.

Therefore, $A = -x^2 + 14x$ has a maximum value.

Example 5

A rectangular box with two square ends has a total surface area of 150 cm^2. Find the dimensions of the box if the volume is a maximum. What is the maximum volume?

Solution:



Let x^2 be the area of a square end and xy be the area of a side.

$$2x^2 + 4xy = 150$$
$$x^2 + 2xy = 75$$
$$y = \frac{75 - x^2}{2x}$$

V (volume) $= x^2 y$

$$= x^2\left(\frac{75 - x^2}{2x}\right)$$
$$= \frac{75x}{2} - \frac{1}{2}x^3$$

$$D_x V = \frac{75}{2} - \frac{3}{2}x^2 = 0$$
$$\frac{3}{2}x^2 = \frac{75}{2}$$
$$x^2 = \frac{75}{3}$$
$$x^2 = 25$$
$$x = 5 \quad (x \text{ cannot be negative.})$$

At $x = 5$, $y = \frac{75 - 5^2}{2(5)}$

$$= \frac{50}{10}$$
$$= 5$$

$$D_x^2 V = -3x$$

At $x = 5$, $D_x^2 V = -15$

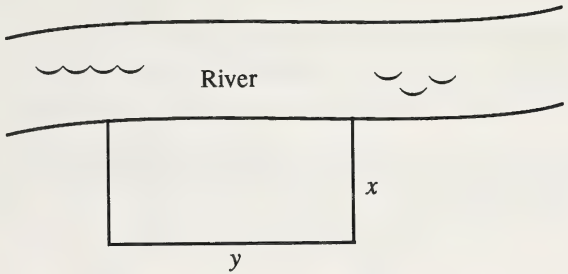
Therefore, $y = 5$ is a maximum.

The dimensions of the box are $5 \text{ cm} \times 5 \text{ cm} \times 5 \text{ cm}$.

The maximum volume is $5 \text{ cm} \times 5 \text{ cm} \times 5 \text{ cm} = 125 \text{ cm}^3$.

Example 6

A farmer wishes to fence in a rectangular field along a straight river as shown in the following diagram. It is not necessary to fence the side bordering the river. The area of the rectangular field is $1800\ \text{m}^2$ and the farmer wishes to use the least length of fencing material. What are the dimensions of the rectangular field?



Solution:

$\text{Area} = xy$

$1800 = xy$

$\therefore y = \dfrac{1800}{x}$

The total length of fence is $l = 2x + y$.

$$l = 2x + \frac{1800}{x}$$

$$D_x l = 2 - 1800x^{-2} = 0$$

$$2 = 1800x^{-2}$$

$$2x^2 = 1800$$

$$x^2 = 900$$

$$x = 30 \qquad (x \text{ cannot be negative.})$$

$$y = \frac{1800}{30}$$

$$= 60$$

$$D_x^2 l = 3600x^{-3}$$

$$= \frac{3600}{x^3}$$

At $x = 30$, $D_x^2 l = \dfrac{3600}{27\,000}$ (positive)

Therefore, l is a minimum.

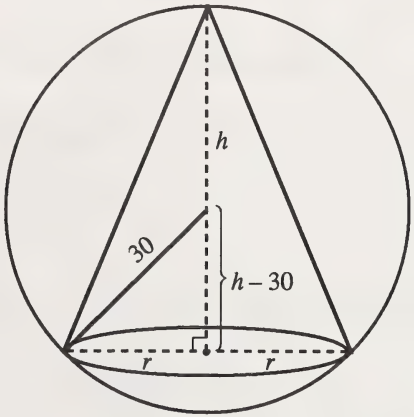
The dimensions of the rectangular field are $30\ \text{m} \times 60\ \text{m}$.

Try a more difficult example.

Example 7

Find the volume of the largest right circular cone that can be inscribed in a sphere of radius 30 cm.

Solution:



Let h be the height of the right circular cone, and let r be the radius of the base of the cone.

$$30^2 = r^2 + (h-30)^2$$

$$900 = r^2 + h^2 - 60h + 900$$

$$r^2 = 60h - h^2$$

$$V = \frac{1}{3}\pi r^2 h$$

$$= \frac{1}{3}\pi\left(60h - h^2\right)h$$

$$= \frac{1}{3}\pi\left(60h^2 - h^3\right)$$

$$D_h V = \frac{1}{3}\pi\left(120h - 3h^2\right) = 0$$

$$h(120 - 3h) = 0$$

$$h = 0, \quad 120 - 3h = 0$$

$$3h = 120$$

$$h = 40$$

Substitute h in the equation for r^2.

$$r^2 = 60h - h^2$$

$$= 60(40) - 40^2$$

$$= 2400 - 1600$$

$$= 800$$

$$V = \frac{1}{3}\pi r^2 h$$

$$= \frac{1}{3}\pi \times 800 \times 40$$

$$= \frac{32\,000}{3}\pi$$

$$\doteq 33\,510.3$$

The volume of the largest right circular cone is $33\,510.3 \text{ cm}^3$.

Now it is your turn to answer some questions.

Do at least the odd-numbered questions.

1. Find the dimensions of a rectangle if the area of the rectangle is $324\ \text{cm}^2$ and the perimeter is a minimum.

2. Find the dimensions of a rectangle if the perimeter is 100 cm and the area of the rectangle is a maximum.

3. A farmer wants to build two adjacent rectangular cattle pens on $60\,000\ \text{m}^2$ of land. The farmer also wants these cattle pens to be of equal area. What is the least amount of fence needed?

4. A rectangular box which is open at the top has a square base. The volume is $171.5\ \text{cm}^3$. If the box is to require the least amount of material, what must its dimensions be?

5. A can has a circular base. It is open at the top and is in the form of a circular cylinder. It has a volume of $8000\pi\ \text{cm}^3$. If the can is to require the least amount of material, what must be its dimensions?

6. If a trapezoid has its longer base on the x-axis and the other two vertices inscribed in $y = 4 - x^2$, find the maximum area of the trapezoid.

7. If an isosceles triangle is inscribed in a circle of radius 4 cm, find the dimensions of the isosceles triangle of maximum area.

8. Find the maximum volume of the largest right circular cone that can be inscribed in a sphere of radius 12 cm.

9. Find the dimensions of the right circular cylinder of largest volume that can be inscribed in a sphere of radius 30 cm.

10. A page contains $600\ \text{cm}^2$. The margins at the top and bottom are 3 cm. The margins at each side are to be 2 cm. What are the dimensions of the paper if the printed area is a maximum?

11. A page contains $150\ \text{cm}^2$ of printed area. The margins at the top and bottom are 3 cm. The margins at each side are 2 cm. What are the dimensions of the smallest such page?

12. Find the dimensions of a rectangle of maximum area inscribed in a circle of radius 6 cm.

For solutions to **Activity 2,** turn to the **Appendix, Topic 1.**

Activity 3

Determine other applications involving maxima and minima.

There are many other applications involving maxima and minima. The businessperson wants to maximize profit and minimize cost. The engineer wants to maximize efficiency in order to determine the maximum height of a projectile. These are examples of using other applications involving minima and maxima. You will continue in this activity by looking at further examples of other minima and maxima problems.

Example 8

The cost of fuel used by a boat varies directly as the cube of the speed of the boat. It costs \$2.50/h when the speed is 5 km/h. If you want to rent this boat, it costs \$40/h plus fuel cost. What is the most economical speed for a voyage of 90 km? What is the minimum cost?

Solution:

Let C = total cost
and v = speed of the boat.

Fuel cost per hour $= kv^3$

$$2.5 = k5^3$$

$$k = \frac{2.5}{125}$$

$$= 0.02$$

The fuel cost per hour is $0.02v^3$.

Total cost = (rent per hour + fuel cost per hour) × time

$$C = \left(40 + 0.02v^3\right)\left(\frac{90}{v}\right)$$

$$= \frac{3600}{v} + 1.8v^2$$

At a minimum,

$$D_v C = -3600v^{-2} + 3.6v = 0$$

$$3.6v = 3600v^{-2}$$

$$3.6v^3 = 3600$$

$$v^3 = 1000$$

$$v = 10$$

At $v = 10$, $C = \frac{3600}{10} + 1.8(10)^2$

$$= 360 + 180$$

$$= 540$$

Therefore, the most economical speed is 10 km/h.

The minimum cost is \$540.

$$\text{Time} = \frac{\text{distance}}{\text{speed}}$$

Example 9

A toy retailer finds that 100 teddy bears can be sold if each is priced at \$12. However, twenty more can be sold for every \$1 decrease in price. The cost of each bear is \$6. In addition the retailer must pay a fixed cost of \$150. How should the bears be priced in order to realize the most profit?

Solution:

Let x = selling price for each bear
and P = profit.

$$\text{Number of bears sold} = 100 + 20(12 - x)$$
$$= 340 - 20x$$

$$\text{Cost} = 150 + (340 - 20x)(6)$$
$$= 2190 - 120x$$

Profit = total sales – cost

$$P = x(340 - 20x) - (2190 - 120x)$$
$$= -20x^2 + 340x - 2190 + 120x$$
$$= -20x^2 + 460x - 2190$$

$$D_x P = -40x + 460 = 0$$
$$x = \frac{460}{40}$$
$$= 11.5$$

At $x = 11.5$,

$$P = -20(11.5)^2 + 460(11.5) - 2190$$
$$= 455$$

Therefore, the bears should be priced at \$11.50 each and the maximum profit is \$455.

Look at another example.

$-20x^2 + 460x - 2190$ is a quadratic function. The coefficient of the x^2 term is negative. Therefore, there is a maximum value of P. (You may want to take the second derivative to verify that P has a maximum value.)

Example 10

The current in a battery is $I = \frac{E}{R+r}$, where E is the electromotive force, R is the external resistance, and r is the internal resistance. E and r are characteristics of the battery and, therefore, are constants. The power developed is $P = RI^2$. If the internal resistance of a battery is 0.15 Ω, what is R when P is a maximum?

Ω = ohm, unit of resistance

Solution:

$$P = RI^2$$

$$= R\left(\frac{E}{R+r}\right)^2$$

$$= \frac{RE^2}{R^2+2rR+r^2}$$

If $r = 0.15\,\Omega$, then $P = \frac{E^2R}{R^2+0.3R+0.0225}$.

$$D_R P = \frac{\left(R^2+0.3R+0.0225\right)D_R\left(E^2R\right)-\left(E^2R\right)D_R\left(R^2+0.3R+0.0225\right)}{\left(R^2+0.3R+0.0225\right)^2}$$

$$= \frac{\left(R^2+0.3R+0.0225\right)\left(E^2\right)-\left(E^2R\right)(2R+0.3)}{\left(R^2+0.3R+0.0225\right)^2}$$

$$= \frac{E^2R^2+0.3RE^2+0.0225E^2-2E^2R^2-0.3E^2R}{\left(R^2+0.3R+0.0225\right)^2}$$

$$D_R P = \frac{-E^2R^2 + 0.0225E^2}{\left(R^2 + 0.3R + 0.0225\right)^2}$$

If $D_R P = 0$, then $-E^2R^2 + 0.0225E^2 = 0.$

$$\therefore E^2R^2 = 0.0225E^2$$

$$R^2 = 0.0225$$

$$R = 0.15$$

$$D_R P = \frac{-E^2R^2 + 0.0225E^2}{\left(R^2 + 0.3R + 0.0225\right)^2}$$

$$= \left(-E^2R^2 + 0.0225E^2\right)\left(R^2 + 0.3R + 0.0225\right)^{-2}$$

$$D_R^2 P = \left(-E^2R^2 + 0.0225E^2\right) D_R\left(R^2 + 0.3R + 0.0225\right)^{-2} + \left(R^2 + 0.3R + 0.0225\right)^{-2} D_R\left(-E^2R^2 + 0.0225E^2\right)$$

$$= \left(-E^2R^2 + 0.0225E^2\right)(-2)\left(R^2 + 0.3R + 0.0225\right)^{-3}(2R + 0.3) + \left(R^2 + 0.3R + 0.0225\right)^{-2}\left(-2E^2R\right)$$

If $R = 0.15$, then $D_R^2 P = 0 - \frac{0.3E^2}{(0.09)^2}$

$$\doteq -37.037E^2 \quad \text{(negative)}$$

Therefore, when $R = r = 0.15\ \Omega$, P is a maximum.

Example 11

When a population of N bacteria is introduced into a nutrient medium, the population grows according to the equation $P = N + \frac{Nt}{81+t^2}$, where t is the number of hours of growth. If $N = 5000$, determine the maximum size of this population and when that size is realized.

Solution:

When $N = 5000$,

$$P = 5000 + \frac{5000t}{81+t^2}$$

$$D_t P = \frac{\left(81+t^2\right)D_t(5000t) - (5000t)D_t\left(81+t^2\right)}{\left(81+t^2\right)^2}$$

$$= \frac{\left(81+t^2\right)(5000) - (5000t)(2t)}{\left(81+t^2\right)^2}$$

$$= \frac{405\,000 + 5000t^2 - 10\,000t^2}{\left(81+t^2\right)^2}$$

$$= \frac{405\,000 - 5000t^2}{\left(81+t^2\right)^2}$$

At a maximum, $D_t P = 0$.

Therefore,

$$405\,000 - 5000t^2 = 0$$

$$5000t^2 = 405\,000$$

$$t^2 = 81$$

$$t = 9 \quad (t \text{ cannot be negative.})$$

When $t = 9$, $P = 5000 + \frac{5000(9)}{81+81}$

$$= 5000 + \frac{45\,000}{162}$$

$$\doteq 5000 + 277.78$$

$$\doteq 5278$$

The maximum size of this population is approximately 5278, and this occurs when $t = 9$ h.

In the quadratic equation $405\,000 - 5000t^2 = 0$, the coefficient of t^2 is negative; thus, it has a maximum.

Example 12

The sum of two natural numbers is 28. Find these two numbers if their product is a maximum.

Solution:

Let x and y be the two natural numbers.

$x + y = 28$

$\therefore\ y = 28 - x$

$$\text{Product } (P) = xy$$
$$= x(28 - x)$$
$$= 28x - x^2$$

At a maximum, $D_x P = 28 - 2x = 0$.

$$\therefore 2x = 28$$
$$x = 14$$

At $x = 14$, $y = 28 - 14$

$$= 14$$

Therefore, the two natural numbers are 14 and 14.

You may wish to view the video titled **Maxima and Minima**. This video is program 6 in the *Catch 31*[1] series. This program provides good reinforcement on solving maxima and minima problems.

Now answer the following questions.

Do at least all the odd-numbered questions.

1. Find two consecutive natural numbers if the sum of the larger number and four times the reciprocal of the smaller number is a minimum.

2. The sum of two natural numbers is 4. If the sum of their cubes is a minimum, what are these numbers?

3. The cruising speed of an airplane is 300 km/h. The formula $C = 250 + \frac{h}{10} + \frac{2\,500\,000}{h}$ represents the cost per hour of flying this airplane, where h is the height in metres. Determine the height at which the cost of flying is a minimum.

This is a quadratic equation.

The coefficient of x^2 is negative; therefore, it has a maximum.

[1] *Catch 31* is a title of ACCESS Network.

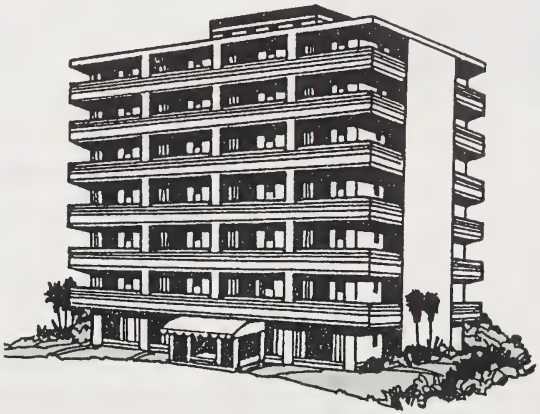
4. The efficiency of a bolt is represented by $E = \frac{\left(m - \frac{1}{3}m^2\right)}{\left(m + \frac{1}{3}\right)}$, where m is the tangent of the pitch angle of the bolt. Determine m for which E is a maximum.

5. The current in a battery is $I = \frac{E}{R+r}$, where E is the electromotive force, R is the external resistance, and r is the internal resistance. E and r are constants. The power developed is $P = I^2 R$. What is R when $r = 0.2\ \Omega$ and the power is a maximum?

6. When a population of N bacteria is introduced into a nutrient medium, the population grows according to the equation $P = N + \frac{Nt}{t^2 + 64}$, where t is the number of hours of growth. If $N = 2000$, determine the maximum size of this population and when that maximum population occurs.

7. An apartment management firm tends an apartment block containing 150 units. All 150 units are rented at \$460 per unit with each unit costing \$72.50/month for utility and repairs. For every \$25 rent increase, four fewer apartments are occupied. What rent should be charged in order to realize the most profit? (Assume that rent would only be increased by a multiple of \$25.)

For solutions to **Activity 3,** turn to the **Appendix, Topic 1.**



If you require help, do the Extra Help section.

If you want more challenging explorations, do the Extensions section. } You may decide to do both.

Extra Help

To solve practical problems involving maxima and minima, you should follow these steps.

Step 1: Express the function to be maximized or minimized as a function of appropriate independent variable(s).

Step 2: If there is more than one independent variable, find the relationships between the independent variables and use those relationships to express the function from Step 1 as a function of a single variable.

Step 3: Take the derivative of the function, and determine its zeros.

Step 4: Use the result from Step 3 to determine the values of other independent variables and possible maximum or minimum values of the function (by substitution).

Step 5: Verify that it is a possible maximum or minimum values using an appropriate test such as the second derivative test.

Example 13

The area of a rectangle is 81 cm^2. Find the dimensions of the rectangle if its perimeter is a minimum. What is the minimum perimeter?

Solution:

Step 1: Let l = length of the rectangle,
w = width of the rectangle,
P = perimeter of the rectangle, and
$P = 2l + 2w$.

Step 2: Area $= l \times w$

$$81 = l \times w$$

$$w = \frac{81}{l}$$

$$P = 2l + 2\left(\frac{81}{l}\right)$$

$$= 2l + 162l^{-1}$$

Step 3: $\frac{dP}{dl} = 2 + (-1)(162)l^{-2}$

$= 2 - 162l^{-2}$

Let $2 - 162l^{-2} = 0$

$2 = 162l^{-2}$

$2l^2 = 162$

$l^2 = 81$

$l = 9$ (l is always positive.)

Step 4: Substitute l into $w = \frac{81}{l}$.

Thus, $w = \frac{81}{9}$ cm $= 9$ cm.

Substitute l and w into $P = 2(l + w)$.

$P = 2(9 + 9)$

$= 36$

Step 5: $\frac{d^2P}{dl^2} = 324l^{-3}$

At $l = 9$, $\frac{d^2P}{dl^2} = \frac{324}{729}$. (positive)

Therefore, $P = 36$ cm is a minimum.

Now try the following questions. If you feel uncomfortable with your solutions, review the examples in this topic.

1. The perimeter of a rectangle is 100 cm. Find the dimensions of the rectangle if its area is a maximum. What is the maximum area?

2. An open box has a square base. The total surface area of all four sides and the base is 48 cm^2. Determine the dimensions of the box if its volume is a maximum. What is the maximum volume?

For solutions to **Extra Help,** turn to the **Appendix, Topic 1.**

Extensions

To determine the maximum or minimum value of a function, you have to find the zeros of its first derivative. Sometimes it is not that easy. If the function is of a degree higher than 3, you may find it very difficult to factor its first derivative. Look at the following example.

Example 14

Find the maximum and/or minimum values of the function $y = 3x^4 - 1x^3 - 6x^2 + 3x + 5$.

Solution:

$$D_x y = 12x^3 - 3x^2 - 12x + 3 = 0$$
$$4x^3 - x^2 - 4x + 1 = 0$$
$$x^2(4x-1) - 1(4x-1) = 0 \quad \text{(Factor.)}$$
$$(x^2 - 1)(4x-1) = 0$$
$$(x-1)(x+1)(4x-1) = 0$$

$x - 1 = 0$, $x + 1 = 0$, $4x - 1 = 0$

$x = 1$ $x = -1$ $x = \frac{1}{4}$

$D_x^2 y = 36x^2 - 6x - 12$

At $x = 1$, $y = 3(1)^4 - (1)^3 - 6(1)^2 + 3(1) + 5$
$= 3 - 1 - 6 + 3 + 5$
$= 4$

$D_x^2 y = 36(1)^2 - 6(1) - 12$
$= 18$ (positive)

Thus, 4 is a minimum value.

At $x = -1$, $y = 3(-1)^4 - (-1)^3 - 6(-1)^2 + 3(-1) + 5$
$= 3 + 1 - 6 - 3 + 5$
$= 0$

$D_x^2 y = 36(-1)^2 - 6(-1) - 12$
$= 36 + 6 - 12$
$= 30$ (positive)

Thus, 0 is a minimum value.

$$\text{At } x=\frac{1}{4},\ y=3\left(\frac{1}{4}\right)^4-\left(\frac{1}{4}\right)^3-6\left(\frac{1}{4}\right)^2+3\left(\frac{1}{4}\right)+5$$

$$=\frac{3}{256}-\frac{1}{64}-\frac{6}{16}+\frac{3}{4}+5$$

$$=\frac{3-4-96+192+1280}{256}$$

$$=\frac{1375}{256}$$

$$=5\frac{95}{256}$$

$$D_x^2 y=36\left(\frac{1}{4}\right)^2-6\left(\frac{1}{4}\right)-12$$

$$=\frac{36}{16}-\frac{6}{4}-12$$

$$=-11\frac{1}{4}\quad\text{(negative)}$$

Thus, $5\frac{95}{256}$ is a maximum value.

Try this question.

Find the maximum and/or minimum values of the function $y=x^4+4x^3-8x^2-48x+4$.

For solutions to **Extensions,** turn to the **Appendix, Topic 1.**

Topic 2 Implicit Differentiation and Motion Problems

Introduction

In Unit 1 you saw one interpretation of the derivative, namely, the slope of a curve. In this topic you are going to consider another interpretation.

Suppose that an object is moving in a straight line. As time changes, distance changes. The ratio of the measure of the distance travelled to the measure of the time taken is called velocity. If you use Δy to represent change in distance and Δt to represent change in time, then velocity is represented by $\frac{\Delta y}{\Delta t}$. When $\Delta t \to 0$, $\frac{\Delta y}{\Delta t}$ becomes the velocity at a given time, and it is also the derivative of y with respect to t. When you read on, you will be able to investigate the relationship between derivative, velocity, and acceleration.

What Lies Ahead

Throughout the topic you will learn to

1. define velocity and solve related problems
2. define acceleration and solve related problems
3. use implicit differentiation to solve motion problems

Now that you know what to expect, turn the page to begin your study of implicit differentiation and motion problems.

Exploring Topic 2

Activity 1

Define velocity and solve related problems.

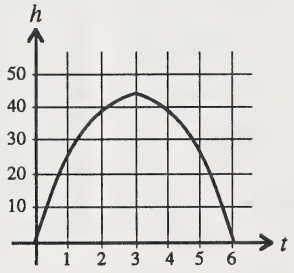
Motion problems are physics problems. Therefore, it is necessary to review some of the basic facts and definitions in physics.

If you move an object along a straight line a distance of 6 cm and then move it backward 2 cm, the total **distance** this object has travelled is 6 cm + 2 cm = 8 cm. The object now is 4 cm from the starting point. This 4 cm is called **displacement.**

In physics, distance travelled per unit time is called **speed** and displacement travelled per unit time is called **velocity.** In the previous example, if you can move that object 8 cm in two seconds, the average speed is $\frac{8 \text{ cm}}{2 \text{ s}} = 4 \text{ cm/s}$. The average velocity is $\frac{4 \text{ cm}}{2 \text{ s}} = 2 \text{ cm/s}$. If you make a round trip between Edmonton and Calgary, you travel a distance of approximately 600 km. If the trip takes six hours, then your average speed is $\frac{600 \text{ km}}{6 \text{ h}} = 100 \text{ km/h}$ and your average velocity is $\frac{\text{displacement}}{\text{time}} = 0 \text{ km/h}$. The term **average speed** is used here because in practice speed and velocity are seldom truly uniform for more than a very short period of time.

If you throw a rock vertically upward from ground level with an initial velocity of 30 m/s, it immediately begins to slow down at the rate of approximately -10 m/s^2 due to gravity $\left(g = -9.8 \text{ m/s}^2\right)$. The rock will stop at some point and then it will fall back toward the ground with its velocity increasing at the same rate. The height of the rock is a function of time defined in physics by $h = -5t^2 + 30t$, where $t \geq 0$, h is height in metres, and t is time in seconds.

The following is the graph of this function.



$h = -5t^2 + 30t$

$s = v_0 t + \frac{1}{2} g t^2$, where
s = displacement
v_0 = initial velocity
t = time
g = acceleration due to gravity

Distance and speed are scalar quantities which do not have direction.

Velocity and displacement are vector quantities which have magnitude and direction.

Use the travelling time for velocity and speed calculations.

According to this function, when $t = 0$, $h = -5(0)^2 + 30(0)$

$= 0$

$t = 1$, $h = -5(1)^2 + 30(1)$

$= 25$

$t = 4$, $h = -5(4)^2 + 30(4)$

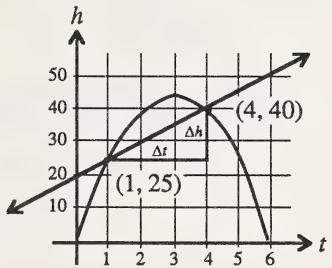
$= 40$

The displacement between $t = 1$ and $t = 4$ is (40 m – 25 m).

The average velocity is $\frac{(40-25)}{4-1} = \frac{15}{3} = 5$ m / s. Use Δh to represent displacement and Δt to represent change in time.

Average velocity $= \frac{\Delta h}{\Delta t}$

Now draw a secant line joining the two points (1, 25) and (4, 40). Δh is the rise of the secant line and Δt is the run of the secant line. The secant slope is $\frac{\Delta h}{\Delta t}$.



$\Delta h = 40 - 25 = 15$

$\Delta t = 4 - 1 = 3$

Therefore, slope of the secant = average velocity = $\frac{\Delta h}{\Delta t}$. Move these two points closer. If the distance between these two points is very small and is approaching zero, then the slope of the secant becomes the slope of the tangent at a point. (Refer to **Unit 1**, First Principles.) The average velocity becomes the velocity at a given time. This velocity is called **instantaneous velocity**. According to this definition, $\lim_{\Delta t \to 0} \frac{\Delta h}{\Delta t}$ = instantaneous velocity = slope of the tangent.

Since $\lim_{\Delta t \to 0} \frac{\Delta h}{\Delta t} = \frac{dh}{dt}$, which is the first derivative of h with respect to t, you can conclude that the velocity of a moving object at any instant is the first derivative with respect to time of the displacement function. It is also the slope of the tangent to the curve of the function.

Go back to considering the motion of the rock. When the rock reaches its highest point, the velocity becomes zero. It stops and then falls back. In other words, $\frac{dh}{dt} = 0$ when h is a maximum (in this case).

Since $h = -5t^2 + 30t$, $\frac{dh}{dt} = -10t + 30 = 0$

$10t = 30$

$t = 3$

When $t = 3$, $h = -5(3)^2 + 30(3)$

$= -45 + 90$

$= 45$

Therefore, the rock reaches its maximum height of 45 m when $t = 3$ s.

Example 1

An airplane flies 1200 km from City *A* to City *B* at 600 km/h and returns at 400 km/h.

- What is the average velocity for the round trip?

Solution:

Total displacement = 0

$$\text{Total time} = \left(\frac{1200}{600} + \frac{1200}{400}\right)\text{h}$$
$$= (2+3)\text{h}$$
$$= 5\text{ h}$$

$$\text{Average velocity} = \frac{0}{5}$$
$$= 0$$

The average velocity is 0 km/h.

- What is the average speed for the round trip?

Solution:

Total time = 5 h

$$\text{Total distance} = 1200 + 1200$$
$$= 2400$$

$$\text{Average speed} = \frac{2400}{5}$$
$$= 480$$

The average speed is 480 km/h.

Example 2

A particle is moving away from point *P* (0, 10). The displacement from a reference point (0, 0) is given by $s = 2t^2 - 8t + 10$, where $t \geq 0$, s is displacement in metres, and t is time in seconds.

- Find the average velocity between $t = 3$ and $t = 5$.

Solution:

$$\text{When } t = 3,\ s = 2(3)^2 - 8(3) + 10$$
$$= 4$$
$$t = 5,\ s = 2(5)^2 - 8(5) + 10$$
$$= 20$$

$$\text{Average velocity} = \frac{\Delta s}{\Delta t}$$
$$= \frac{20-4}{5-3}$$
$$= \frac{16}{2}$$
$$= 8$$

The average velocity between $t = 3$ and $t = 5$ is 8 m/s.

- Find the velocity at any time *t*.

Solution:

The velocity at any time is equal to the first derivative of the function.

$$D_t s = 4t - 8,\ t \geq 0$$

- Find the velocity at $t = 6$.

Solution:

$$\text{At } t = 6,\ D_t s = 4(6) - 8$$
$$= 16$$

The velocity at $t = 6$ is 16 m/s.

- Find the minimum value of s.

Solution:

$$\text{Let } 4t - 8 = 0$$
$$4t = 8$$
$$t = 2$$

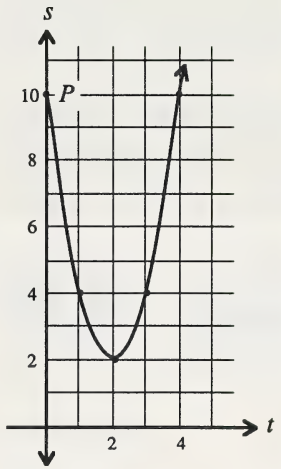
$$\text{At } t = 2,\ s = 2(2)^2 - 8(2) + 10$$
$$= 8 - 16 + 10$$
$$= 2$$

Therefore, the minimum value of s is 2 m.

- Sketch the graph of $s = 2t^2 - 8t + 10$ and describe the motion of the particle.

Solution:

t	s
4	10
3	4
2	2
1	4
0	10



The particle falls from a height of 10 m until it reaches a height of 2 m. It then reverses its direction and goes up. You can see that the particle starts out at a point P 10 m from the reference point (0, 0) and begins to move toward the reference point. When the displacement between the particle and the reference point is 2 m, the particle reverses its direction and moves away from the reference point.

$D_t^2 s = 4$; therefore, $s = 2$ m is a minimum.

Now do at least all the odd-numbered questions in the following exercise.

1. City A is 150 km from City B, and City B is 250 km from City C. The three cities are in a straight line as shown. A car leaves City A at 8:00 a.m. and arrives at City B at 9:30 a.m. After a coffee break the same car leaves City B at 10:00 a.m. It arrives at City C at 12:30 p.m. It leaves City C at 1:00 p.m. and returns to City B at 4:00 p.m.

   a. What is the car's displacement?
   b. What is the average velocity?
   c. What is the average speed?

2. A car is driven along a straight highway for three hours at 90 km/h and then for four hours at 100 km/h.

   a. What is its displacement?
   b. What is the average velocity?

3. An object is moving upward away from a fixed point. The position (in metres) of the object with respect to the fixed point is a function of time (in seconds) given by $h = -5t^2 + 45t + 25$, where $t \geq 0$.

   a. What is the velocity at any time?
   b. What is the velocity at $t = 1$?
   c. Find t when the object reaches its maximum displacement.
   d. What is its maximum displacement?
   e. What is the average velocity from $t = 0$ to $t = 2$?

4. An object is moving in a straight line from a fixed point. The displacement in metres which is a function of time (t) in seconds is given by $s = t^2 - 10t + 25$.

   a. Find the average velocity from $t = 1$ to $t = 4$.
   b. What is the velocity at any time?
   c. What is the velocity at $t = 3$?
   d. Find t when s is a minimum.
   e. What is the minimum s?

For solutions to **Activity 1,** turn to the **Appendix, Topic 2.**

Activity 2

Define acceleration and solve related problems.

Suppose that you are driving a car at 50 km/h. The car in front of you is only driving at a speed of 40 km/h and you wish to pass. So you step on the gas to increase the velocity of your car to 80 km/h. You have accelerated your car. Acceleration is the rate at which the velocity changes per unit time. If you increase the velocity of your car from 50 km/h to 80 km/h in three seconds, then the acceleration is $\frac{80-50}{3}$ km/h per second = 10 km/h per second. It can be written as 10 km/h/s. **Acceleration is rate of change of velocity.**

$$\therefore \text{Average acceleration} = \frac{\text{corresponding change in velocity}}{\text{change in time}}$$

$$= \frac{v_2 - v_1}{t_2 - t_1}$$

$$= \frac{\Delta v}{\Delta t}$$

Again, make the change in time (Δt) very small. When Δt approaches zero, the acceleration becomes the acceleration at a given time. It is called **instantaneous acceleration** and is denoted by the symbol a.

Thus, $a = \lim\limits_{\Delta t \to 0} \frac{\Delta v}{\Delta t}$.

Since $\lim\limits_{\Delta t \to 0} \frac{\Delta v}{\Delta t}$ is the derivative of v with respect to t,

$a = \frac{dv}{dt}$ or $a = D_t v$.

But $v = D_t s$.

Therefore, $a = D_t (D_t s)$, which is the second derivative of s with respect to t.

$D_t (D_t s)$ can be written as $D_t^2 s \left(\text{or } \frac{d^2 s}{dt^2} \right)$.

Therefore, $a = D_t^2 s = D_t v$.

How do you take the second derivative of a function? (Review **Unit 2.**) You take the derivative of the function. Then consider the result as a new function and take the derivative of the result.

Caution: Notice that the expression **with respect to t** is emphasized and the symbol D_t is used. Always specify the variable being **differentiated**. (Differentiation is the process of taking the derivative.) D standing alone with no variable specified is meaningless. $D_t (3t^2) = 6t$, but $D_x (3t^2)$ cannot be calculated unless there is some way of expressing t in terms of x.

D_t^2 does not mean **D_t squared**. It means **take the second derivative of.**

Look at the following examples.

Example 3

The velocity of an object in metres per second is given by $v = 3t^2 - 5t + 8$, where $t \geq 0$. Find the acceleration at $t = 3$.

Solution:

$$v = 3t^2 - 5t + 8$$

$$a = D_t v = 6t - 5$$

$$\text{At } t = 3, \; a = 6(3) - 5$$

$$= 18 - 5$$

$$= 13$$

The acceleration of the object is 13 m/s^2 when $t = 3$.

The next example begins with a distance equation.

Example 4

A particle is moving in a straight line. The distance from a fixed point is given by $s = t^3 - 5t^2 + 7t + 9$, where $t \geq 0$.

- Find the velocity and acceleration at any time.

Solution:

$s = t^3 - 5t^2 + 7t + 9, \quad t \geq 0$

$v = D_t s = 3t^2 - 10t + 7, \quad t \geq 0$

$a = D_t v = D_t^2 s = 6t - 10, \quad t \geq 0$

Therefore, the velocity at any time is $3t^2 - 10t + 7$.
The acceleration at any time is $6t - 10$.

- Determine the turning points, that is, the points where the velocity is zero.

Solution:

$3t^2 - 10t + 7 = 0$

$(3t - 7)(t - 1) = 0$

$t = \frac{7}{3}, \quad t = 1$

At $t = 1$, $s = 12$.

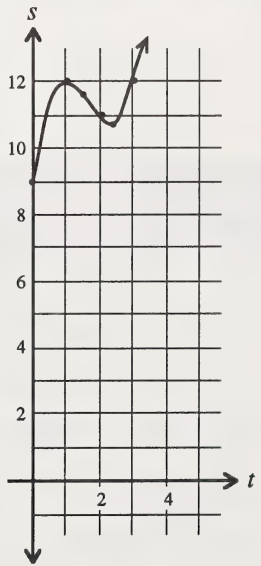
At $t = \frac{7}{3}$, $s = 10\frac{22}{27}$

Therefore, (1, 12) and $\left(\frac{7}{3}, 10\frac{22}{27}\right)$ are the turning points.

- Sketch the graph of $s = t^3 - 5t^2 + 7t + 9$.

Solution:

t	0	1	$1\frac{1}{2}$	2	$2\frac{1}{3}$	3
s	9	12	$11\frac{5}{8}$	11	$10\frac{22}{27}$	12



- Describe the direction of the motion at $t = 0, 1, 2, 2\frac{1}{3}$, and 3 s.

Solution:

$$v = 3t^2 - 10t + 7$$

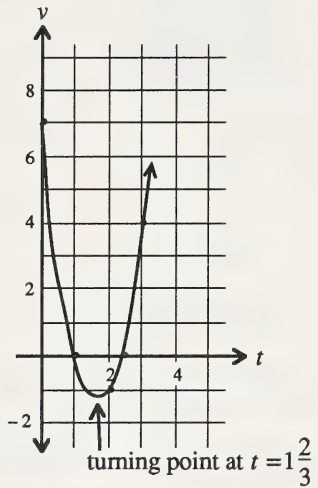
At $t = 0$, $v = 0 + 0 + 7 = 7$	Therefore, the direction of motion is positive.
At $t = 1$, $v = 3 - 10 + 7 = 0$	Therefore, the particle stops.
At $t = 2$, $v = 3(2)^2 - 10(2) + 7 = -1$	Therefore, the direction of motion is negative (moving back toward the starting point).
At $t = 2\frac{1}{3}$, $v = 3\left(\frac{7}{3}\right)^2 - 10\left(\frac{7}{3}\right) + 7 = 0$	The particle stops.
At $t = 3$, $v = 3(3)^2 - 10(3) + 7 = 4$	The direction of motion is positive.

- Sketch the graphs of v versus t and a versus t. Describe both v and a.

Solution:

$$v = 3t^2 - 10t + 7$$

t	0	1	2	$2\frac{1}{3}$	3
v	7	0	-1	0	4

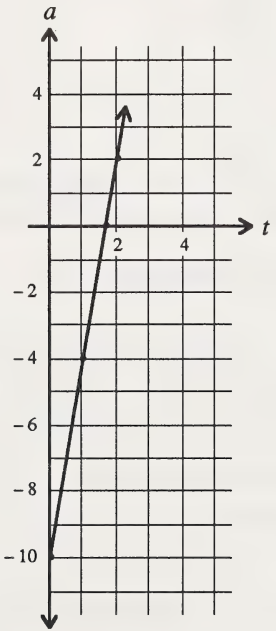


$$D_t v = 6t - 10 = 0$$
$$t = \frac{10}{6}$$
$$= 1\frac{2}{3}$$

The velocity is decreasing from $t = 0$ to $t = 1$. At $t = 1$ the particle stops and the velocity is zero. From $t > 1$ to $t < 2\frac{1}{3}$ the particle is moving in the opposite direction. It stops again when $t = 2\frac{1}{3}$. The particle turns around and moves in a positive direction at $t > 2\frac{1}{3}$. The velocity is positive and increasing after that.

$a = 6t - 10$

t	0	1	$1\frac{2}{3}$	2
a	-10	-4	0	2



The acceleration is increasing from negative to positive, or, for example, from deceleration to acceleration at a constant rate. When a is negative, the velocity decreases. When a is positive, the velocity increases.

Sometimes more complex functions can be used to define a velocity or displacement function, but the basic principles remain the same.

Example 5

A particle moves in a straight line. The displacement in metres is given by

$$s = \frac{t^2}{2+t}, \text{ where } t \geq 0.$$

Determine the acceleration at the instant when the velocity is $\frac{3}{4}$ m/s.

Solution:

If $s = \frac{t^2}{2+t}$, where $t \geq 0$,

$$\text{then } v = D_t s = \frac{(2+t)D_t\left(t^2\right) - \left(t^2\right)D_t(2+t)}{(2+t)^2} \quad \text{(quotient rule)}$$

$$= \frac{(2+t)(2t) - t^2}{(2+t)^2}$$

$$= \frac{4t + 2t^2 - t^2}{(2+t)^2}$$

$$= \frac{t^2 + 4t}{(2+t)^2},$$

and $a = D_t v = \dfrac{(2+t)^2 D_t\left(t^2+4t\right) - \left(t^2+4t\right) D_t (2+t)^2}{(2+t)^4}$ (quotient rule)

$$= \frac{(2+t)^2(2t+4) - \left(t^2+4t\right)(2)(2+t)}{(2+t)^4}$$

$$= \frac{(2+t)^2(2)(2+t) - 2(2+t)\left(t^2+4t\right)}{(2+t)^4}$$

$$= \frac{2(2+t)\left[\left(4+4t+t^2\right) - \left(t^2+4t\right)\right]}{(2+t)^4} \quad \text{(Factor.)}$$

$$= \frac{2(2+t)(4)}{(2+t)^4}$$

$$= \frac{8}{(2+t)^3}$$

You are given $v = \frac{3}{4}$ m / s.

Therefore, $\dfrac{t^2+4t}{(2+t)^2} = \dfrac{3}{4}$

$$4\left(t^2+4t\right) = 3(2+t)^2$$

$$4t^2 + 16t = 12 + 12t + 3t^2$$

$$t^2 + 4t - 12 = 0$$

$$(t+6)(t-2) = 0$$

$$t+6 = 0, \quad t-2 = 0$$

$$t = -6 \qquad t = 2$$

Since $t \geq 0$, $t \neq -6$.

At $t = 2$, $a = \dfrac{8}{(2+t)^3}$

$$= \frac{8}{(2+2)^3}$$

$$= \frac{1}{8}$$

The acceleration of the particle is $\frac{1}{8}$ m / s^2 when the velocity is $\frac{3}{4}$ m / s.

If you have access to a videocassette player, you may find it helpful to view the video titled **Motion: Distance, Velocity, and Acceleration**. This video is program 7 in the *Catch 31*[1] series. This program should help you understand the applications of velocity and acceleration.

[1] *Catch 31* is a title of ACCESS Network.

Now do at least the odd-numbered questions in the following exercise.

In each of questions 1 to 4 the velocity of a particle is given. Find the acceleration at any time and at the particular times required.

1. $v = 2t - 3t^2$
   Find a when $t = 4$.

2. $v = (t+3)^2$
   Find a when $t = 3$.

3. $v = 5t^3 - 2t$
   Find a when $t = 2$.

4. $v = \dfrac{10t}{3+t^2}$
   Find a when $t = 1$.

5. An object is moving in a straight line. The distance in metres from a fixed point is given by $s = t^2 - 5t + 4$, where $t \geq 0$.

   a. Find the velocity and acceleration at any time.
   b. Determine the turning point, that is, the point where the velocity is zero.
   c. Sketch the graph of the function $s(t)$.
   d. Describe the direction of the motion at $t = 1$, $\frac{5}{2}$, and 3.
   e. Sketch the graph of v versus t and describe the velocity.
   f. Sketch the graph of a versus t and describe the acceleration.

6. An object is moving in a straight line. The distance in metres from a fixed point is given by $s = 2t^3 - 15t^2 + 24t + 8$, where $t \geq 0$.

   a. Find the velocity and acceleration at any time.
   b. Determine the turning points, that is, the points where the velocity is zero.
   c. Sketch the graph of the function $s(t)$.
   d. Describe the direction of the motion at $t = 0$, 1, 3, 4, and 5.
   e. Sketch the graph of v as a function of t and describe the velocity.
   f. Sketch the graph of a as a function of t and describe the acceleration.

7. A particle is moving in a straight line. The distance in metres from a fixed point is given by $s = 3t^3 - 3t^2 + t + 9$, where $t \geq 0$. Determine the acceleration when the velocity is 4 m/s.

8. A particle is moving in a straight line. The distance in metres from a fixed point is given by $s = \frac{t^2}{3+t}$, where $t \geq 0$. Determine the acceleration when the velocity is $\frac{3}{4}$ m/s.

For solutions to **Activity 2,** turn to the **Appendix, Topic 2.**

Activity 3

Use implicit differentiation to solve motion problems.

In the last section you dealt with motion of particles given their displacement or velocity as a function of time.

In certain instances a relationship between displacement and velocity may be given by an equation like $3s^2 + 5v^2 = 28$.

What is the acceleration in this case?

Relations like this and others can be dealt with using an adaptation of the chain rule called **implicit differentiation**. The method of implicit differentiation was discussed in **Unit 2, Topic 1, Activity 6**.

Look at an example.

Example 6

If $3s^2 + 5v^2 = 28$, where s is displacement and v is velocity, find the acceleration (a).

Solution:

Realize that s, v, and a are all functions of t. Consequently, you can differentiate any of them with respect to t.

Begin by taking the derivative of each term with respect to t.

$$\frac{d(3s^2)}{dt} + \frac{d(5v^2)}{dt} = \frac{d(28)}{dt}$$

But $\frac{d(3s^2)}{dt} = \frac{d(3s^2)}{ds} \times \frac{ds}{dt}$ (by the chain rule)

$$= 6s \times \frac{ds}{dt},$$

and $\frac{d(5v^2)}{dt} = \frac{d(5v^2)}{dv} \times \frac{dv}{dt}$ (by the chain rule)

$$= 10v \times \frac{dv}{dt}.$$

Therefore, $\frac{d(3s^2)}{dt} + \frac{d(5v^2)}{dt} = \frac{d(28)}{dt}$ becomes

$$6s\frac{ds}{dt} + 10v\frac{dv}{dt} = 0.$$

Since $\frac{ds}{dt} = v$ and $\frac{dv}{dt} = a$, $6sv + 10va = 0$

$$v(6s + 10a) = 0$$

$$6s + 10a = 0$$

$$a = -\frac{6s}{10}$$

$$= -\frac{3s}{5}$$

For additional help on this activity you may wish to view the video titled **Functions Defined Implicitly**. This video is program 8 in the *Catch 31*[1] series.

Now you can use implicit differentiation to solve some other motion problems.

Example 7

The relation between the velocity and distance is given by $5v^2 = 40s + 200$, where s is the displacement from a fixed point and v is the velocity of a moving object. Find the acceleration.

Solution:

$$5v^2 = 40s + 200$$
$$D_t\left(5v^2\right) = D_t(40s) + D_t(200)$$
$$10v\, D_t v = 40\, D_t s + 0$$

Since $D_t v = a$, $D_t s = v$.

Therefore, $10va = 40v$

$$a = \frac{40v}{10v}$$
$$= 4$$

When taking the derivative of an equation, it is important to know which symbol represents a constant.

[1] *Catch 31* is a title of ACCESS Network.

Example 8

An object is dropped from a window. The total energy (E), in joules (J), is given by $E = \frac{1}{2}mv^2 + mgs$, where m is the mass of the object in kilograms, g is the acceleration due to gravity $\left(-9.8 \text{ m/s}^2\right)$, v is the velocity in metres per second, and s is the distance from the ground in metres. If E is a constant, find the acceleration (a).

Solution:

$$\frac{d(E)}{dt} = \frac{d\left(\frac{1}{2}mv^2\right)}{dt} + \frac{d(mgs)}{dt}$$

Since E is a constant, $\frac{dE}{dt} = 0$.
Since m is the mass of the object, it is a constant.
Since g is always equal to -9.8 m/s^2, it is a constant.

Hence, $0 = \frac{1}{2}m\frac{dv^2}{dt} + mg\frac{ds}{dt}$

$$0 = \frac{1}{2}m(2v)\frac{dv}{dt} + mg\frac{ds}{dt}$$
$$= mv(a) + mg(v)$$

$$a = -\frac{mgv}{mv}$$
$$= -g$$

Now it is practice time. Do at least the first two questions.

1. An object is moving away from a fixed point. The relation between the velocity and the distance is given by $3v^2 = 18s + 300$, where s is the distance from a fixed point and v is the velocity. Find the acceleration.

2. An object is moving away from a fixed point. It has a mass of m kg and has moved a distance of s m. Its velocity is v. The total energy is given by $E = \frac{1}{2}mv^2 + mps$. If E, m, and p are constants, find the acceleration (a).

3. A gun is fired. The bullet has travelled a distance (s) in the barrel of the gun. The velocity of the bullet is given by $v = \frac{6000s}{6+s}$. Find the acceleration in terms of s.

4. The relation between the velocity (v) and time (t) is given by $\frac{1}{v} + \frac{1}{3} = 4t$. Prove that the acceleration is $-4v^2$.

5. Find $\frac{dy}{dx}$ if $\sqrt{(y^2 - 2)} = xy$.

6. Find $D_x y$ if $x^3 y - 4y^2 = 3$.

For solutions to **Activity 3,** turn to the **Appendix, Topic 2.**

If you require help, do the Extra Help section.

If you want more challenging explorations, do the Extensions section. } You may decide to do both.

Extra Help

If s = distance,
t = time,
v = velocity,
a = acceleration,

and they are in appropriate units, then $v = \frac{ds}{dt}$ and $a = \frac{dv}{dt} = \frac{d^2s}{dt^2}$.

When velocity is zero, the moving object reaches its local maximum or local minimum. (Velocity is equal to the slope of the tangent to the s-t curve. The curve represents the distance of the object from a fixed point.) When $v = 0$, the tangent is a horizontal line which indicates that a local maximum or a local minimum exists. When acceleration is zero, the moving object reaches its maximum or minimum velocity. If $a = 0$, v does not change temporarily. The object is still moving at a constant velocity. The velocity of an object at any point is the instantaneous velocity of the object at that point. The instantaneous velocity is $\frac{ds}{dt}$.

Speed and velocity are different.

$$\text{Average speed} = \frac{\text{total distance travelled}}{\text{elapsed time}}$$

$$\text{Average velocity} = \frac{\text{displacement}}{\text{elapsed time}}$$

The direction of motion is determined by the direction of the velocity. If the velocity is positive, then the direction of the motion is positive. If the velocity is negative, then the direction of the motion is negative. If the velocity is zero, the motion stops.

If $s = t^3 - 6t^2 + 9t + 24$, where $t \geq 0$, then the instantaneous velocity at any time is $v = \frac{ds}{dt} = 3t^2 - 12t + 9$.

The instantaneous velocity at $t = 2$ is as follows:

$$v = 3(2)^2 - 12(2) + 9$$
$$= -3 \qquad \text{(moving toward the starting point)}$$

The acceleration at any time is $a = \frac{dv}{dt} = 6t - 12$.

The acceleration at $t = 3$ is $a = 6(3) - 12 = 6$.

The average velocity in the interval from $t = 2$ to $t = 5$ is as follows:

$$\text{At } t = 2,\ s = 2^3 - 6(2)^2 + 9(2) + 24$$
$$= 8 - 24 + 18 + 24$$
$$= 26$$

$$\text{At } t = 5,\ s = 5^3 - 6(5)^2 + 9(5) + 24$$
$$= 125 - 150 + 45 + 24$$
$$= 44$$

$$\text{Average velocity} = \frac{44 - 26}{5 - 2}$$
$$= \frac{18}{3}$$
$$= 6$$

The distance has a maximum or minimum when $\frac{ds}{dt} = v = 0$.

$$3t^2 - 12t + 9 = 0$$
$$t^2 - 4t + 3 = 0$$
$$(t-1)(t-3) = 0$$
$$t - 1 = 0,\quad t - 3 = 0$$
$$t = 1 \qquad t = 3$$

$$\text{At } t = 1,\ s = 1^3 - 6(1)^2 + 9(1) + 24$$
$$= 28$$
$$a = \frac{d^2 s}{dt^2} = 6(1) - 12$$
$$= -6 \quad \text{(negative)}$$

Therefore, 28 is a maximum distance.

$$\text{At } t = 3,\ s = 3^3 - 6(3)^2 + 9(3) + 24$$
$$= 24$$
$$a = \frac{d^2 s}{dt^2} = 6(3) - 12$$
$$= 6 \quad \text{(positive)}$$

Therefore, 24 is a minimum distance.

The velocity has a maximum or minimum when $\frac{dv}{dt} = 0$.

$$a = \frac{dv}{dt} = 6t - 12 = 0$$
$$6t = 12$$
$$t = 2$$

$$\text{At } t = 2,\ v = 3(2)^2 - 12(2) + 9$$
$$= 12 - 24 + 9$$
$$= -3$$
$$\frac{d^2 v}{dt^2} = 6 \quad \text{(positive)}$$

Therefore, -3 is a minimum velocity.

Implicit differentiation can be used to solve motion problems. If a relation between s and v is given and you want to find the acceleration, which is $\frac{dv}{dt}$, implicit differentiation is the tool to use.

Example 9

If $v^2 = 4s^2 + 200$, where v is the velocity of a moving object and s is the distance from a fixed point, find the acceleration at any time.

Solution:

$$v^2 = 4s^2 + 200$$
$$D_t\left(v^2\right) = D_t\left(4s^2\right) + D_t(200)$$
$$2v\,D_t v = 8s\,D_t s + 0$$

Since $D_t v = a$ and $D_t s = v$, $2va = 8sv$

$$a = \frac{8sv}{2v}$$
$$= 4s$$

Do the following questions.

1. If $s = 2t^3 - 21t^2 + 60t + 80$, where s is the distance in metres and t is the time in seconds, find the following:

   a. the instantaneous velocity at $t = 2$
   b. the instantaneous acceleration at $t = 3$
   c. the average velocity in the interval from $t = 1$ to $t = 3$
   d. the maximum or minimum velocity (if any)

2. If $x^2 + 2xy + 4y^2 = 3$, find $\frac{dy}{dx}$.

3. If $3v^2 = 5s + 1000$, where v is the velocity of a moving object and s is the distance from a fixed point, find the acceleration at any time.

For solutions to **Extra Help,** turn to the **Appendix, Topic 2.**

Extensions

Sometimes you have to substitute the original equation to meet the conditions of the question.

Example 10

The velocity (v) of a wind-up toy, in metres per second, is given by $v = \frac{300s}{3+s}$ where s is the length of the spring. Find the acceleration in terms of the length s.

Solution:

$$D_t v = \frac{(3+s)D_t(300s) - (300s)D_t(3+s)}{(3+s)^2}$$

$$D_t v = \frac{(3+s)(300D_t s) - (300s)D_t s}{(3+s)^2}$$

Since $D_t s = v$ and $D_t v = a$,

$$a = \frac{(3+s)(300v) - 300sv}{(3+s)^2}.$$

By factoring you get $a = \frac{300v(3+s-s)}{(3+s)^2}$

$$= \frac{900v}{(3+s)^2}.$$

Since $v = \frac{300s}{3+s}$ (from the original equation),

$$a = \frac{900}{(3+s)^2} \times \frac{300s}{(3+s)}$$

$$= \frac{270\,000s}{(3+s)^3}.$$

Now try the following.

1. If $s^2 + 25v^2 = 100$ defines a relation between distance (s) and velocity (v), find a relation between acceleration (a) and distance (s).

2. The velocity (v) is given by $v = \frac{1-3s^2}{3s^2-2}$ where s is the displacement. Find the acceleration (a) in terms of s.

For solutions to **Extensions,** turn to the **Appendix, Topic 2.**

Topic 3 Rates of Change

Introduction

Have you ever put oil in the engine of a car using a cone-shaped funnel? If you have, you have probably observed that as the oil drains, the level of the oil in the funnel appears to go down at an ever increasing rate. The rate at which the level falls is dependent on the depth in the funnel **and** the rate at which the volume is decreasing. The rate of change of volume and depth are **related rates.**

In this topic you are going to solve related-rate problems.

What Lies Ahead

Throughout the topic you will learn to

1. solve applied problems involving rates of change

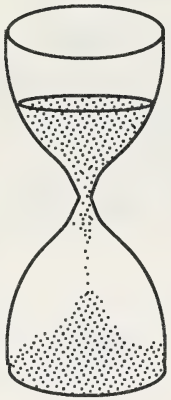
Now that you know what to expect, turn the page to begin your study of rates of change.

Exploring Topic 3

Activity 1

Solve applied problems involving rates of change.

You will recall that in the introduction to this topic you noted that when oil is drained through a cone-shaped funnel, the level appears to go down more quickly as the oil drains. When you think about it, this means that the rate at which the oil level is falling is a function of the depth of the oil in the funnel. It is also related to the rate of change of the volume of the oil, since you can increase the rate of fall by making a larger outlet in the funnel. So the rate at which the level of the oil falls is **jointly** dependent on the depth of the oil and the rate at which the volume is changing.

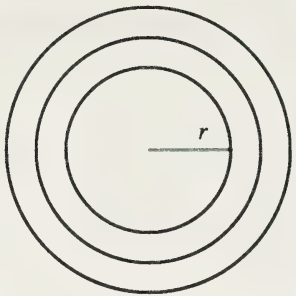


Similar observations can be made by watching sand fall through an egg timer. The rate of change of the volume is constant; only so many grains of sand can fit through the opening at any one time. Despite this, the level falls more quickly as the sand runs out so the rate of fall is dependent not only on the rate of change of volume, but also on the depth of the sand in the top of the timer.

In this section you will consider problems in which rates of change of some variables are related.

Relations in mathematics refer to a correspondence between two or more variables. This correspondence or rule is often expressed as an equation.

For example, $C = 2\pi r$ expresses the relationship between the circumference of a circle and its radius. If the radius of a circle is increasing at a certain rate, what happens to the circumference of that circle? The circumference of the circle is also increasing.



The rate of change of a variable implies a change **with respect to time**. You have already dealt with questions that involve the rate of change of displacement (velocity), and the rate of change of velocity (acceleration). In that discussion the rates of change at any time were determined by differentiating **with respect to time**. If you do the same here, you will find a relationship between the rates of change of the radius and the circumference of the circle.

Given $C = 2\pi r$ and the fact that the radius is changing as time changes, you can use implicit differentiation **with respect to time** to determine the relationship between their rates of change.

Given $C = 2\pi r$, you differentiate with respect to time.

So, $\frac{d}{dt}(C) = \frac{d}{dt}(2\pi r)$.

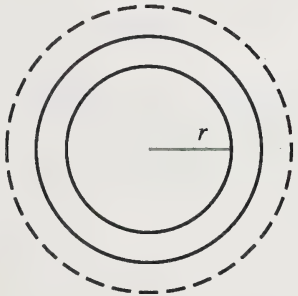
That is, $\frac{dC}{dt} = 2\pi \frac{dr}{dt}$.

The relationship indicates that the rate of change of circumference **with respect to time** is equal to 2π times the rate of change of the radius **with respect to time**.

This is an example of the concept of **related rates.**

Example 1

A circular ring is heated so that it expands. If the rate of increase of the radius is 0.01 cm/s, determine the rate at which the circumference is increasing.



Solution:

The circumference of a circle is related to the radius by the following formula:

$$C = 2\pi r$$

Differentiating with respect to time, you get the following:

$$\frac{d}{dt}(C) = \frac{d}{dt}(2\pi r)$$
$$\frac{dC}{dt} = 2\pi \frac{dr}{dt}$$

You are given that the rate of change of the radius is 0.01 cm/s. Hence, $\frac{dr}{dt} = 0.01$ cm/s. Since it is expanding, the rate of change is positive.

$$\frac{dC}{dt} = 2\pi \frac{dr}{dt}$$
$$\frac{dC}{dt} = 2\pi(0.01)$$
$$\frac{dC}{dt} = 0.02\pi$$

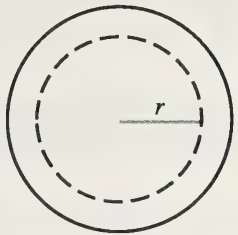
The circumference of the ring is increasing at the rate of 0.02π cm/s.

The preceding example presents a relatively basic application of the concept of related rates.

Now look at a more complex example.

Example 2

A circular disc is heated and then cooled. During the cooling process, the radius is found to be decreasing at the rate of 0.02 mm/s. At what rate is the area of the disc changing when the radius of the disc is 100 mm?



Solution:

The area of the disc and its radius are related by the following formula:

$$A = \pi r^2$$

Differentiating with respect to time, you get the following:

$$\frac{d}{dt}(A) = \frac{d}{dt}\left(\pi r^2\right)$$

$$= \frac{d}{dr}\left(\pi r^2\right)\frac{dr}{dt} \quad \text{(chain rule)}$$

That is, $\frac{dA}{dt} = 2\pi r \frac{dr}{dt}$.

You are given $\frac{dr}{dt} = -0.02$ mm/s and you are interested in the rate of change of area when $r = 100$ mm.

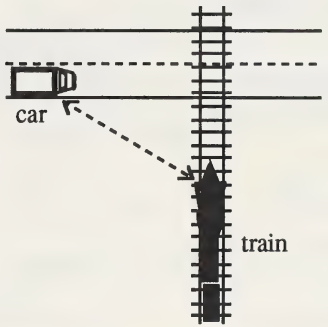
Substituting these values, you get the following:

$$\frac{dA}{dt} = 2\pi(100 \text{ mm})(-0.02 \text{ mm/s})$$

$$\frac{dA}{dt} = -4\pi \text{ mm}^2/\text{s}$$

The area is decreasing (the sign of $\frac{dA}{dt}$ is negative) at the rate of 4π mm^2/s.

You have probably been dealing with related rates subconsciously for a long time. If you are in a car approaching a railroad crossing at the same time that a train is approaching, what considerations do you automatically make?



As time increases, radius decreases; thus, $\frac{dr}{dt}$ is negative.

The most important thing to most drivers in this situation is the rate at which the distance between the car and the train is changing. This rate of change is dependent on both the rate at which your car is travelling and the rate at which the train is travelling. It is also affected by the distances of the car and the train from the crossing.

Consider a specific example of this scenario.

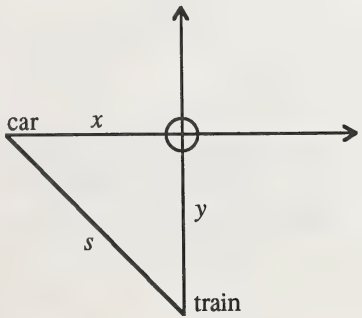
Example 3

A straight level road crosses a railroad track. This crossing forms right angles. A car is on the road 1 km from the crossing travelling at 80 km/h toward the crossing. At the same time, a train is 1.2 km from the crossing and approaching at 95 km/h.

At what rate is the distance between the car and the train changing at that instant?

Solution:

The diagram shows the relative positions of the train and the car as they approach the crossing.



Let x = distance between the car and the crossing,
y = the distance between the train and the crossing, and
s = the distance between the car and the train.

These distances are related by the Pythagorean theorem.

$$s^2 = x^2 + y^2$$

Each of these variables is a function of time, so you can differentiate with respect to time.

$$\frac{d}{dt}\left(s^2\right) = \frac{d}{dt}\left(x^2\right) + \frac{d}{dt}\left(y^2\right)$$

$$2s\frac{ds}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt}$$

Now you know $\frac{dx}{dt} = -80$ km / h and $\frac{dy}{dt} = -95$ km / h since both the car and the train are moving toward the crossing; thus, their distances from the crossing are decreasing.

You are also given that $x = 1$ km and $y = 1.2$ km, so you can find s using $s^2 = x^2 + y^2$.

$$s^2 = x^2 + y^2$$
$$s^2 = 1^2 + 1.2^2$$
$$= 1 + 1.44$$
$$= 2.44$$
$$s \doteq 1.56$$

If you substitute into the differentiated equation, you get the following:

$$2(1.56)\frac{ds}{dt} \doteq 2(1)(-80)+2(1.2)(-95)$$

$$3.12\frac{ds}{dt} \doteq -388$$

$$\frac{ds}{dt} \doteq -124.4$$

At this point the distance between the train and the car is decreasing at the rate of 124.4 km/h.

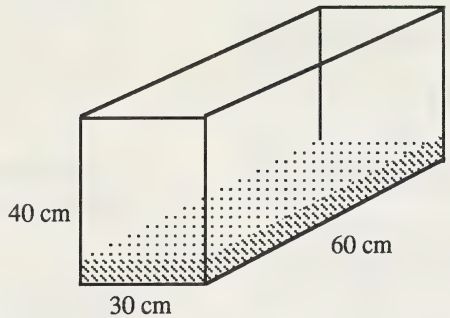
A method for dealing with these types of problems is suggested by the previous examples.

- Determine a relationship between the variable(s) for which the rate(s) of change is (are) given and the variable for which the rate of change is to be found.
- Differentiate that equation with respect to time.
- Substitute the values given (or determined by using the given values) and solve for the desired rate.

Apply this procedure to another example.

Example 4

Water is being poured into a fish aquarium that is 60 cm long, 30 cm wide, and 40 cm deep. Determine a relationship between the rate at which water is being poured and the rate at which the depth is increasing.



Solution:

You are looking for a relation between the rate of change of **volume of water** and the rate of change of **depth of water**. As water is poured into the tank, it takes the shape of the container.

So, $V_{\text{water}} = l_{\text{water}} \times w_{\text{water}} \times h_{\text{water}}$.

Of these, the only one that varies is h_{water}.

The 40 cm is not used in this problem other than as a limit for h.

$\therefore V = 60 \times 30 \times h$

$V = 1800h$ (relates the volume of water to the depth of the water)

Differentiating with respect to time, you get $\frac{dV}{dt} = 1800\frac{dh}{dt}$.

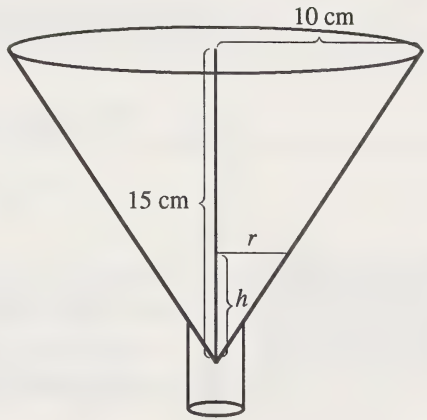
This is the relationship between the rates of change of volume and depth of water.

This example shows the importance of identifying variables and constants in the problem. If a quantity remains constant throughout the problem, its value can be substituted **before** you differentiate. If the quantity is variable, you cannot substitute any value given for it until **after** you have differentiated.

Now think about oil being drained through a cone-shaped funnel. The rate at which the level of the oil in the funnel appears to go down can be found as shown in the next example.

Example 5

Engine oil is added to the engine of a car. The funnel used has a radius of 10 cm and the height of the cone-shaped portion is 15 cm. (See the following diagram.) The oil is drained from the funnel at a rate of 12 cm^3 / s. When the oil level is 5 cm from the bottom, find the rate at which the oil level is falling.



Solution:

Let V = volume of oil in cubic centimetres at time t,
h = height of the oil level in centimetres, and
r = radius of the oil surface in centimetres.

V, h, and r are variables that change as time changes.

$V = \frac{1}{3}\pi r^2 h$ (volume of a cone)

The answer you want is $\frac{dh}{dt}$.

Since there are three variables, you have to get rid of r by finding a relationship between h and r. Look at the triangles in the diagram; according to the properties of similar triangles, you can say $\frac{r}{h} = \frac{10}{15}$.

Therefore, $r = \frac{2}{3}h$.

Substitute r into V.

$$V = \frac{1}{3}\pi\left(\frac{2}{3}h\right)^2 h$$
$$= \frac{1}{3}\pi\left(\frac{4}{9}h^2\right)h$$
$$= \frac{4}{27}\pi h^3$$

Differentiate with respect to t.

$$\frac{dV}{dt} = \frac{4}{27}\pi\frac{d\left(h^3\right)}{dt}$$
$$= \frac{4}{27}\pi\left(3h^2\right)\frac{dh}{dt}$$

Since $\frac{dV}{dt} = 12 \text{ cm}^3/\text{s}$,

at $h = 5$ cm, $12 \text{ cm}^3/\text{s} = \frac{4\pi}{27}(3)(5 \text{ cm})^2 \frac{dh}{dt}$.

$$\frac{dh}{dt} = \frac{(12)(9)}{4\pi(25)} \text{ cm/s}$$
$$= \frac{27}{25\pi} \text{ cm/s}$$
$$\doteq 0.34 \text{ cm/s}$$

Therefore, the oil level is falling at $\frac{27}{25\pi}$ cm/s when the oil is 5 cm deep.

For additional examples of related-rate problems, you may wish to view the video titled **Related Rates**. This video is program 9 in the *Catch 31*[1] series.

Do at least the first eight questions.

1. The area of a circle is $9\pi \text{ cm}^2$ and is increasing at the rate of $\frac{1}{2}\pi \text{ cm}^2/\text{min}$. Find the rate at which the radius of the circle is increasing.

2. If you are moving a 2 m post away from a light source 6 m above the ground at a rate of 1.5 m/s, at what rate is the end of the shadow of the post moving away horizontally?

3. A 3 m post is leaning against a vertical wall. If the bottom of the post begins to slip away from the wall at a rate of 0.5 m/s, how fast is the top of the post moving down the wall when it is 1 m from the ground?

4. Coal is poured onto the top of a conical pile at the rate of $8 \text{ m}^3/\text{min}$. The diameter of the cone is three times the height. How fast is the height increasing when the pile is 4 m high?

[1] *Catch 31* is a title of ACCESS Network.

5. A trough has ends which are isosceles trapezoids. It is 50 cm long, 20 cm high, 30 cm wide at the top, and 10 cm wide at the bottom. If it is filled with water at a rate of $600 \text{ cm}^3/\text{s}$, how fast is the depth increasing when it is 10 cm deep?

6. A cylindrical tank has a radius of 2 m. Water is pumped in at the rate of $3 \text{ m}^3/\text{min}$. How fast is the water level rising when the water is 4 m deep?

7. A tank which has the shape of an inverted cone with a height of 6 m and a radius of 5 m is being filled with water at the rate of $3\pi \text{ m}^3/\text{min}$. How fast is the water level rising when the water is 3 m deep?

8. Helium is pumped into a spherical balloon at the rate of $100 \text{ cm}^3/\text{s}$. How fast is the radius increasing when the radius is 5 cm?

9. A trough has two ends which are equilateral triangles. It is 9 m long and 2 m wide at the top. Water is pumped in at the rate of $3 \text{ m}^3/\text{min}$. How fast does the water level rise when the deepest point is 0.5 m?

10. The curve $y = 2x^2$ is the path of a moving point. If the x-coordinate of this point is increasing at the rate of four units per second when $x = 3$, at what rate is the y-coordinate changing?

For solutions to **Activity 1,** turn to the **Appendix, Topic 3.**

In an isosceles trapezoid the nonparallel sides are of equal length.

If you require help, do the Extra Help section.

If you want more challenging explorations, do the Extensions section. } You may decide to do both.

Extra Help

In a related-rate problem all the rates are derivatives with respect to time. To solve a problem, you need to do the following:

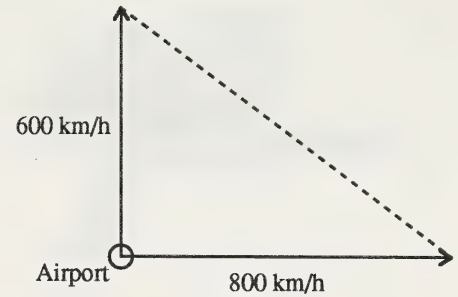
- List all the rates involved in the problem.
- Find an equation relating them and make sure that the equation is true within the time interval being considered.
- Differentiate the equation with respect to time.
- Substitute the given values into the result of your differentiation to find the desired quantity.

Look at the following example.

Example 6

Two airplanes start out from the same airport at the same time. One flies east at 800 km/h, and the other flies north at 600 km/h. How fast are they separating after two hours?

Solution:



Let the velocity of the first airplane be 800 km/h and the distance travelled by this airplane at any time be x. Let the velocity of the second airplane be 600 km/h and the distance travelled by this airplane at any time be y. Let s be the distance between the two airplanes.

After 2 h, $x = 2 \times 800 = 1600$ and $y = 2 \times 600 = 1200$.

$$\frac{dx}{dt} = 800 \text{ km / h}$$

$$\frac{dy}{dt} = 600 \text{ km / h}$$

$$s^2 = x^2 + y^2$$
$$s^2 = 1600^2 + 1200^2$$
$$= 2\,560\,000 + 1\,440\,000$$
$$= 4\,000\,000$$
$$\therefore s = 2000$$

$$D_t\left(s^2\right) = D_t\left(x^2\right) + D_t\left(y^2\right)$$
$$2s\,D_t s = 2x\,D_t x + 2y\,D_t y$$
$$2(2000)\left(D_t s\right) = 2(1600)(800) + 2(1200)(600)$$
$$4000\,D_t s = 2\,560\,000 + 1\,440\,000$$
$$D_t s = \frac{4\,000\,000}{4000}$$
$$= 1000$$

Therefore, the two airplanes are separating at 1000 km/h.

Do either of the following questions.

1. Two cars start out from the same place at the same time. One car travels west at 50 km/h, and the other travels south at 80 km/h. How fast are they separating after 2 h?

2. A rectangular container is 3 m long, 2 m wide, and 2 m deep. Water is being pumped into it at $1.5\ m^3$ / min. How fast is the surface rising?

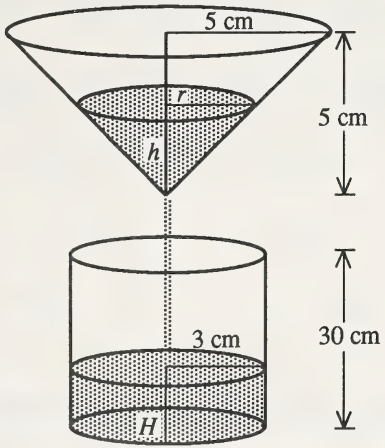
For solutions to **Extra Help**, turn to the **Appendix, Topic 3.**

Extensions

There are many situations in science and engineering that involve the rate of change of some property or quantity. It is impossible to have all kinds of rate problems included in this topic. The following question is different and may be more difficult.

Example 7

Water is being poured through a conical filter into a cylindrical container as shown in the diagram. The filter has a radius of 5 cm and is 5 cm high. The cylinder has a radius of 3 cm and is 30 cm high. The water level in the filter is falling at $\frac{1}{2}$ cm / s when the water in the filter is 4 cm deep. How fast is the water level in the cylinder rising at that instant?



Solution:

The volume of a cone is $V = \frac{1}{3}\pi r^2 h$.

$$\frac{r}{h} = \frac{5}{5}$$
$$= 1$$
$$\therefore r = h$$

$$V = \frac{1}{3}\pi r^2 h$$
$$= \frac{1}{3}\pi h^2 h$$
$$= \frac{1}{3}\pi h^3$$
$$\frac{dV}{dt} = \frac{1}{3}\pi\left(3h^2\right)\frac{dh}{dt}$$

When $h = 4$ cm, $\frac{dh}{dt} = 0.5$ cm / s.

$$\therefore \frac{dV}{dt} = \pi(4)^2(0.5)$$
$$= 8\pi$$

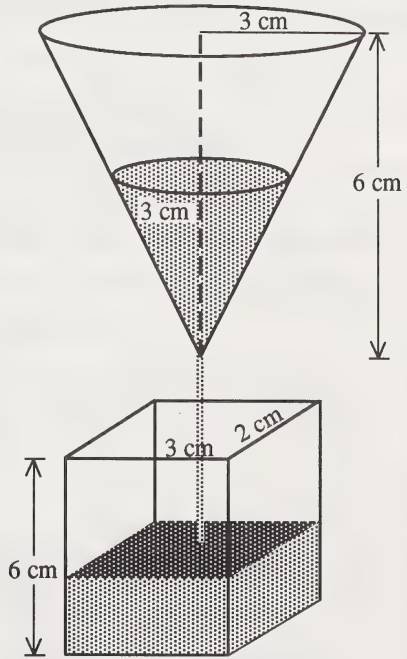
This is the rate of flow out of the filter. Therefore, it must be the rate of flow into the cylinder.

For the cylinder,

$$V = \pi r^2 H$$
$$= \pi (3)^2 H$$
$$= 9\pi H$$

$$\frac{dV}{dt} = 9\pi \frac{dH}{dt}$$
$$\frac{dV}{dt} = 8\pi$$
$$8\pi = 9\pi \frac{dH}{dt}$$
$$\frac{dH}{dt} = \frac{8\pi}{9\pi}$$
$$= \frac{8}{9}$$

The water level is rising at $\frac{8}{9}$ cm/s.



Do the following question.

Water is being poured through a conical filter into a rectangular container as shown in the diagram. The filter has a radius of 3 cm and is 6 cm high. The rectangular container is 3 cm long, 2 cm wide, and 6 cm high. At the instant that the water level in the filter is falling at $\frac{1}{2}$ cm / s, the water in the filter is 3 cm deep, and the water level in the rectangular container is 2 cm deep. How fast is the water level in the rectangular container rising at that instant?

For solutions to **Extensions,** turn to the **Appendix, Topic 3.**

Unit Summary

What You Have Learned

In this unit you developed procedures for dealing with practical problems which can be solved using derivatives. Specifically, you know the following:

- Problems in maxima and minima are solved by
  - determining a function and defining the quantity to be maximized or minimized
  - setting the derivative of that function to zero and solving
  - applying the result from the previous point to answer the question posed
- If $s = f(t)$ is a function defining displacement in terms of time, then the following is true:

$$v = \frac{ds}{dt} \qquad \text{and} \qquad a = \frac{dv}{dt}$$

$$v = f'(t) \qquad\qquad a = \frac{d^2 s}{dt^2}$$

$$a = f''(t).$$

Unit Summary

- Relations can be differentiated implicitly.
- Problems involving related rates are solved by
  - determining a relationship between the variables for which rates of change are given and the variable for which rate of change is to be determined
  - differentiating implicitly **with respect to time**
  - substituting given information in this differentiated form to solve the problem

This is your final unit on differential calculus, but there is still much that you have not investigated. If you have gained an understanding of the concepts discussed to this point, you have a good chance of being successful in your further studies of calculus.

You are now ready to

complete the **Unit Assignment**.

Appendix

Solutions

Review

Topic 1 Maxima and Minima

Topic 2 Implicit Differentiation and Motion Problems

Topic 3 Rates of Change

Review

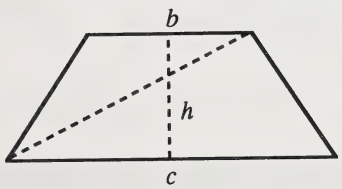
1. a. $A = \pi r^2$, where A = area of the circle, and
r = radius of the circle.

b. $A = \frac{1}{2}lh$, where A = area of the triangle,
l = length of one side, and
h = perpendicular distance between that side and the opposite vertex.

c. $A = lw$, where A = area of the rectangle,
l = length of the rectangle, and
w = width of the rectangle.

d. $A = bh$, where A = area of the parallelogram,
b = length of one side (usually the base), and
h = height or distance across the parallelogram.

e. $A = \frac{1}{2}h(b+c)$



If you divide the trapezoid along a diagonal, you get two triangles. The area of one is $\frac{1}{2}ch$ and the area of the other is $\frac{1}{2}bh$. The total area is $\frac{1}{2}bh + \frac{1}{2}ch$. Removing a common factor, you get $\frac{1}{2}h(b+c)$.

2. $C = 2\pi r$, where C is the circumference of the circle and r is the radius of the circle.

$C = \pi d$, where d is the diameter of the circle.

3. The perimeter of a polygon is the sum of the lengths of its sides.

4.

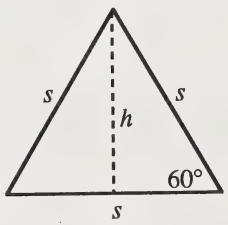


Figure 1

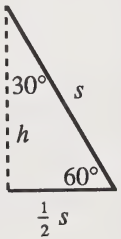


Figure 2

In Figure 1 the area would be given by $A = \frac{1}{2}sh$. Figure 2 shows half of the equilateral triangle. This is a 30° - 60° - 90° triangle and the sides are in the ratio of $1:\sqrt{3}:2$. This means that the side opposite the 60° angle is $\sqrt{3}$ times the side opposite the 30° angle.

Therefore, $h = \sqrt{3} \times \frac{1}{2}s = \frac{\sqrt{3}}{2}s$.

Substitute $h = \frac{\sqrt{3}}{2}s$ in $A = \frac{1}{2}sh$.

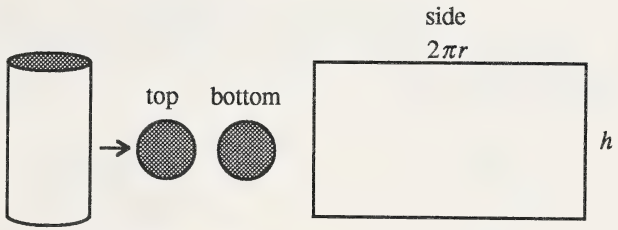
$$A = \frac{1}{2}s\left(\frac{\sqrt{3}}{2}s\right)$$

$$= \frac{\sqrt{3}}{4}s^2$$

5. For a sphere, volume is given by $V = \frac{4}{3}\pi r^3$ and surface area is given by $A = 4\pi r^2$.

   Given that $V = \frac{4}{3}\pi r^3$, it follows that $\frac{dV}{dr} = 4\pi r^2$ which is the formula for surface area. The sphere is the shape that encloses maximum volume for minimum surface area.

6. a. The surface area of a figure is the sum of the areas of its surfaces.



$$\pi r^2 + \pi r^2 + 2\pi rh$$

If a cylinder is taken apart, there is a circular top and bottom and a rectangle which is h units by $2\pi r$ units. This rectangle appears when you cut down the side wall and open it up. The circumference of the circle is $2\pi r$.

$$A = 2\pi r^2 + 2\pi rh$$

The volume of **right-walled** (the walls are at right angles to the base) containers like cylinders and cubes can be found by multiplying the area of the base by the height of the walls.

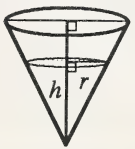
The volume formula for a cylinder is $V = \pi r^2 h$.

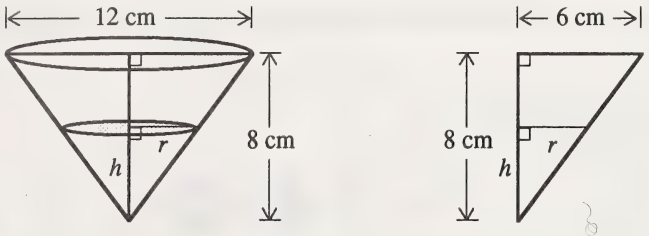
b. Since the cylinder is open-topped, there is only one circle in the surface area formula.

$$A = \pi r^2 + 2\pi rh$$

The volume is still the same, that is, $V = \pi r^2 h$.

7. Since water takes the shape of its container, the water will also be in the shape of a cone and the volume will be given by $V = \frac{1}{3}\pi r^2 h$. A cone contains $\frac{1}{3}$ the volume of a cylinder with the same radius and depth.



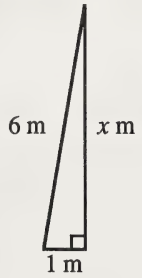
8. 

The diagrams show the cone and water and the similar triangles that can be formed using the two radii and heights. In similar triangles, corresponding sides are proportional.

Thus, $\frac{r}{h}=\frac{6}{8}$

$r=\frac{3}{4}h$

9. The diagram shows the conditions stated in the problem. In any right triangle the square of the hypotenuse equals the sum of the squares of the other two sides.



Thus, $6^2=1^2+x^2$

$36=1+x^2$

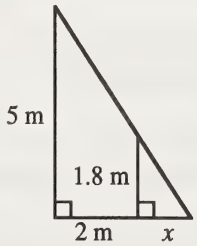
$35=x^2$

$x=\sqrt{35}$

$x\doteq 5.9$

The ladder will reach a point on the wall approximately 5.9 m above the ground.

10. The diagram shows the conditions in the problem.



Let x = length of the shadow.

$\frac{5}{2+x}=\frac{1.8}{x}$

$5x=3.6+1.8x$

$3.2x=3.6$

$x=\frac{9}{8}$

$x=1\frac{1}{8}$

The shadow will be $1\frac{1}{8}$ m long.

If you feel comfortable with this review, you should be prepared to continue with the unit. You might want to summarize the formulas for future reference.

Exploring Topic 1

Activity 1

Determine the turning point and second derivative.

1. $y = 3x^2 - 2x + 9$

$$\frac{dy}{dx} = 6x - 2$$

Let $\frac{dy}{dx} = 0$.

$$6x - 2 = 0$$
$$6x = 2$$
$$x = \frac{2}{6}$$
$$= \frac{1}{3}$$

At $x = \frac{1}{3}$, $y = 3\left(\frac{1}{3}\right)^2 - 2\left(\frac{1}{3}\right) + 9$

$$= \frac{1}{3} - \frac{2}{3} + 9$$
$$= 8\frac{2}{3}$$

$$\frac{d^2y}{dx^2} = 6 \quad \text{(positive)}$$

$\left(\frac{d^2y}{dx^2}\right.$ is another notation for the second derivative.$\left.\right)$

Therefore, $y = 8\frac{2}{3}$ is a minimum.

2. $y = x^3 - 48x + 7$

$$\frac{dy}{dx} = 3x^2 - 48 = 0$$
$$3x^2 = 48$$
$$x^2 = 16$$
$$x = \pm 4$$

At $x = 4$, $y = (4)^3 - 48(4) + 7$

$$= 64 - 192 + 7$$
$$= -121$$

At $x = -4$, $y = -64 + 192 + 7$

$$= 135$$

$$\frac{d^2y}{dx^2} = 6x$$

At $x = 4$, $\dfrac{d^2y}{dx^2} = 6(4)$

$= 24$ (positive)

Therefore, $y = -121$ is a minimum.

At $x = -4$, $\dfrac{d^2y}{dx^2} = 6(-4)$

$= -24$ (negative)

Therefore, $y = 135$ is a maximum.

3. $y = -\dfrac{1}{4}(x-2)^2$

$= -\dfrac{1}{4}\left(x^2 - 4x + 4\right)$

$\dfrac{dy}{dx} = -\dfrac{1}{4}(2x-4) = 0$

$2x = 4$

$x = 2$

At $x = 2$, $y = -\dfrac{1}{4}(2-2)^2$

$= 0$

$\dfrac{d^2y}{dx^2} = -\dfrac{1}{4}(2)$

$= -\dfrac{1}{2}$ (negative)

Therefore, $y = 0$ is a maximum.

4. $y = 2x^3 - 4x^2 + 2x$

$D_x y = 6x^2 - 8x + 2 = 0$

$2\left(3x^2 - 4x + 1\right) = 0$

$2(x-1)(3x-1) = 0$

$x - 1 = 0, \quad 3x - 1 = 0$

$x = 1 \qquad x = \dfrac{1}{3}$

At $x = 1$, $y = 2(1)^3 - 4(1)^2 + 2(1)$

$= 0$

At $x = \dfrac{1}{3}$, $y = 2\left(\dfrac{1}{3}\right)^3 - 4\left(\dfrac{1}{3}\right)^2 + 2\left(\dfrac{1}{3}\right)$

$= \dfrac{2}{27} - \dfrac{4}{9} + \dfrac{2}{3}$

$= \dfrac{8}{27}$

$D_x^2 y = 12x - 8$

At $x = 1$, $D_x^2 y = 12(1) - 8$

$= 4$ (positive)

Therefore, $y = 0$ is a minimum value.

At $x = \frac{1}{3}$, $D_x^2 y = 12\left(\frac{1}{3}\right) - 8$

$= 4 - 8$

$= -4$ (negative)

Therefore, $y = \frac{8}{27}$ is a maximum.

5. $y = x^3 - 2x^2 - 4x - 8$

$D_x y = 3x^2 - 4x - 4 = 0$

$(x-2)(3x+2) = 0$

$x - 2 = 0, \quad 3x + 2 = 0$

$x = 2 \qquad x = -\frac{2}{3}$

At $x = 2$, $y = (2)^3 - 2(2)^2 - 4(2) - 8$

$= 8 - 8 - 8 - 8$

$= -16$

At $x = -\frac{2}{3}$, $y = \left(-\frac{2}{3}\right)^3 - 2\left(-\frac{2}{3}\right)^2 - 4\left(-\frac{2}{3}\right) - 8$

$= -\frac{8}{27} - \frac{8}{9} + \frac{8}{3} - 8$

$= -\frac{176}{27}$

$= -6\frac{14}{27}$

$D_x^2 y = 6x - 4$

At $x = 2$, $D_x^2 y = 6(2) - 4$

$= 8$ (positive)

Therefore, $y = -16$ is a minimum.

At $x = -\frac{2}{3}$, $D_x^2 y = 6\left(-\frac{2}{3}\right) - 4$

$= -8$ (negative)

Therefore, $y = -6\frac{14}{27}$ is a maximum.

6. $y = 3x^3 - 6x^2 + 24x$

$D_x y = 9x^2 - 12x + 24 = 0$

$3(3x^2 - 4x + 8) = 0$

$$x = \frac{4 \pm \sqrt{(-4)^2 - 4(3)(8)}}{2(3)}$$

$$= \frac{4 \pm \sqrt{-80}}{6}$$

Since $\sqrt{-80}$ is not a real number, there is no maximum or minimum for real values of x.

Activity 2

Determine the maxima and minima in geometrical problems.

1. Area of a rectangle $(A) = l \times w$

$324 = l \times w$

$\therefore w = \frac{324}{l}$

Perimeter of a rectangle $(P) = 2l + 2w$

$= 2l + 2\left(\frac{324}{l}\right)$

$= 2l + 648l^{-1}$

$\frac{dP}{dl} = 2 - \frac{648}{l^2} = 0$

$\frac{648}{l^2} = 2$

$l^2 = \frac{648}{2}$

$= 324$

$l = \sqrt{324}$ (positive only)

$= 18$

$w = \frac{324}{18}$

$= 18$

$\frac{d^2P}{dl^2} = 1296l^{-3}$

At $l = 18$, $\frac{d^2P}{dl^2}$ is positive.

Therefore, when $l = 18$ and $w = 18$, the perimeter is a minimum.

2. Perimeter $= 2l + 2w$

$$100 = 2l + 2w$$
$$2w = 100 - 2l$$
$$w = \frac{100 - 2l}{2}$$
$$= 50 - l$$

Area $(A) = l \times w$

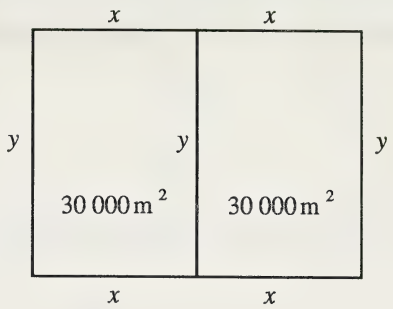
$$= l(50 - l)$$
$$= 50l - l^2$$

$$\frac{dA}{dl} = 50 - 2l = 0$$
$$2l = 50$$
$$l = 25$$
$$w = 50 - 25$$
$$= 25$$

$$\frac{d^2A}{dl^2} = -2 \text{ (negative)}$$

Therefore, the dimensions of the rectangle are 25 cm by 25 cm when the area is a maximum.

3.



Area $= 2xy$

$$60\,000 \text{ m}^2 = 2xy$$
$$\therefore y = \frac{60\,000 \text{ m}^2}{2x}$$
$$= \frac{30\,000 \text{ m}^2}{x}$$

Let P be the amount of fence.

$$P = 3y + 4x$$
$$= 3\left(\frac{30\,000}{x}\right) + 4x$$
$$= \frac{90\,000}{x} + 4x$$
$$\frac{dP}{dx} = -\frac{90\,000}{x^2} + 4 = 0$$
$$4x^2 = 90\,000$$
$$x^2 = 22\,500$$
$$x = 150$$

When $x = 150$, $y = \dfrac{30\,000}{150}$

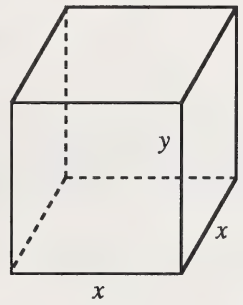
$= 200$

$\dfrac{d^2P}{dx^2} = \dfrac{180\,000}{x^3}$

At $x = 150$, $\dfrac{d^2P}{dx^2}$ is positive.

Therefore, P is a minimum when $x = 150$ and $y = 200$.
The least amount of fence is $3(200) + 4(150) = 1200$ m.

4.



$V = x^2 y$

$171.5 = x^2 y$

$y = \dfrac{171.5}{x^2}$

Let M be the amount of material.

Total surface area $= 4xy + x^2$

$= 4x\left(\dfrac{171.5}{x^2}\right) + x^2$

$= \dfrac{686}{x} + x^2$

$\dfrac{dM}{dx} = -\dfrac{686}{x^2} + 2x = 0$

$2x = \dfrac{686}{x^2}$

$x^3 = 343$

$x = 7$

$y = \dfrac{171.5}{49}$

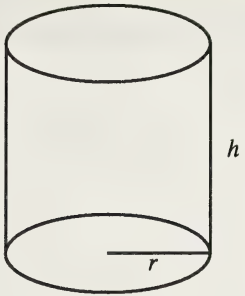
$= 3.5$

$\dfrac{d^2M}{dx^2} = \dfrac{1372}{x^3} + 2$

At $x = 7$, $\dfrac{d^2M}{dx^2}$ is positive; therefore, it is a minimum.

If the box is to require the least amount of material, the dimensions of the box are 7 cm × 7 cm × 3.5 cm.

5.



$$V = \pi r^2 h$$

$$8000\pi = \pi r^2 h$$

$$\therefore h = \frac{8000\pi}{\pi r^2}$$

$$\text{Area } (A) = \pi r^2 + 2\pi rh$$

$$= \pi r^2 + 2\pi r\left(\frac{8000\pi}{\pi r^2}\right)$$

$$= \pi r^2 + \frac{16\,000\pi}{r}$$

$$\frac{dA}{dr} = 2\pi r - 16\,000\pi r^{-2} = 0$$

$$2\pi r = 16\,000\pi r^{-2}$$

$$r^3 = \frac{16\,000\pi}{2\pi}$$

$$= 8000$$

Therefore, $r = 20$ cm.

When $r = 20$, $h = \frac{8000}{r^2}$

$$= \frac{8000}{400}$$

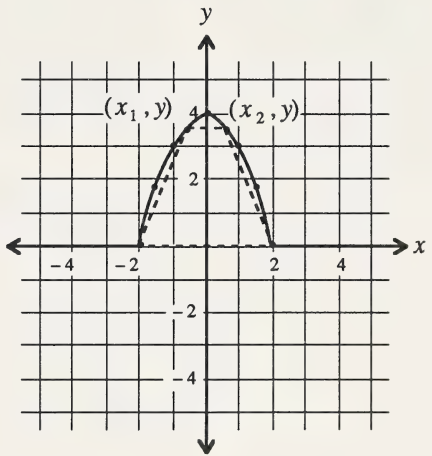
$$= 20$$

$$\frac{d^2A}{dr^2} = 2\pi + \frac{32\,000\pi}{r^3}$$

At $r = 20$, $\frac{d^2A}{dr^2}$ is positive.

Therefore, if the can is to require the least amount of material, its radius should be 20 cm and its height should be 20 cm.

6.



$$y = 4 - x^2$$

The x-intercepts are as follows:

$$4-x^2=0$$
$$x^2=4$$
$$x=\pm 2$$

Therefore, the length of the longer base is $2+2=4$ units.

The height of the trapezoid is y and the length of the shorter base is $2x$.

$$\text{Area } (A)=\frac{1}{2}y(4+2x)$$

Since $y=4-x^2$, $A=\frac{1}{2}(4+2x)\left(4-x^2\right)$

$$=(2+x)\left(4-x^2\right)$$
$$=8+4x-2x^2-x^3$$

$$\frac{dA}{dx}=4-4x-3x^2=0$$
$$3x^2+4x-4=0$$
$$(x+2)(3x-2)=0$$

$$x+2=0,\quad 3x-2=0$$
$$x=-2 \qquad x=\frac{2}{3}$$

If $x=-2$, the area = 0; thus, x cannot be -2.

If $x=\frac{2}{3}$, then $y=4-\left(\frac{2}{3}\right)^2$.

$$=4-\frac{4}{9}$$
$$=\frac{32}{9} \text{ or } 3\frac{5}{9}$$

$$\text{Area}=\frac{1}{2}\left[4+\left(2\times\frac{2}{3}\right)\right]\times\frac{32}{9}$$

$$=\frac{1}{2}\left(5\frac{1}{3}\right)\left(\frac{32}{9}\right)$$

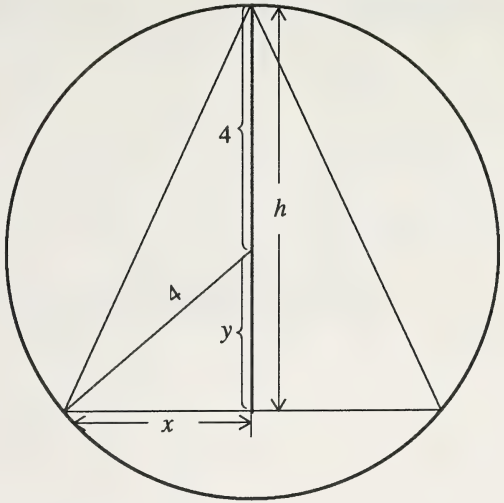
$$=\frac{256}{27} \text{ or } 9\frac{13}{27}$$

$$\frac{d^2A}{dx^2}=-6x-4$$

$$\text{At } x=\frac{2}{3},\ \frac{d^2A}{dx^2}=-6\left(\frac{2}{3}\right)-4$$
$$=-8 \quad \text{(negative)}$$

Therefore, $9\frac{13}{27}$ is a maximum area.

7.



$x^2+y^2=4^2$

$y=h-4$

$x^2=16-y^2$

$=16-(h-4)^2$

$=16-\left(h^2-8h+16\right)$

$=-h^2+8h$

$x=\sqrt{8h-h^2}$

Area of the triangle $(A)=x\bullet h$

$=\sqrt{8h-h^2}\bullet h$

$$D_x A=h\bullet\frac{1}{2}\left(8h-h^2\right)^{-\frac{1}{2}}(8-2h)+\left(8h-h^2\right)^{\frac{1}{2}}=0$$

$$\frac{8h-2h^2}{2\left(8h-h^2\right)^{\frac{1}{2}}}+\frac{\left(8h-h^2\right)^{\frac{1}{2}}}{1}=0$$

$$\frac{8h-2h^2+2\left(8h-h^2\right)}{2\left(8h-h^2\right)^{\frac{1}{2}}}=0$$

$$8h-2h^2+16h-2h^2=0$$

$$-4h^2+24h=0$$

$$h(-4h+24)=0$$

$h=0$ (Discard.), $-4h+24=0$

$h=6$

At $h=6$, $x=\sqrt{8\times6-6^2}$

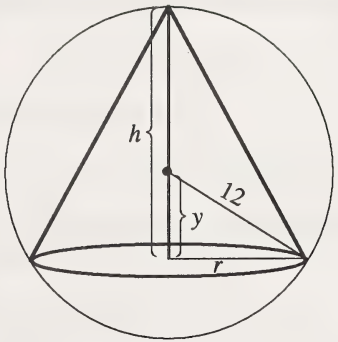
$=\sqrt{12}$

$=2\sqrt{3}$

Therefore, the base is $2x=4\sqrt{3}$.

The height of the triangle is 6 cm and the base is $4\sqrt{3}$ cm when the area is a maximum.

8.



$y^2 + r^2 = 12^2$

$y = h - 12$

$r^2 = 12^2 - y^2$

$= 144 - (h-12)^2$

$= 144 - h^2 + 24h - 144$

$= 24h - h^2$

$r = \sqrt{24h - h^2}$

Volume of a circular cone $(V) = \frac{1}{3}\pi r^2 h$

$= \frac{1}{3}\pi\left(\sqrt{24h - h^2}\right)^2 h$

$= \frac{1}{3}\pi\left(24h - h^2\right)h$

$= \frac{\pi}{3}\left(24h^2 - h^3\right)$

$D_h V = \frac{\pi}{3}\left(48h - 3h^2\right) = 0$

$48h - 3h^2 = 0$

$h(48 - 3h) = 0$

$h = 0$ (Discard.), $48 - 3h = 0$

$3h = 48$

$h = 16$

$r = \sqrt{24h - h^2}$; thus, $r^2 = 24(16) - 16^2$

$r^2 = 128$

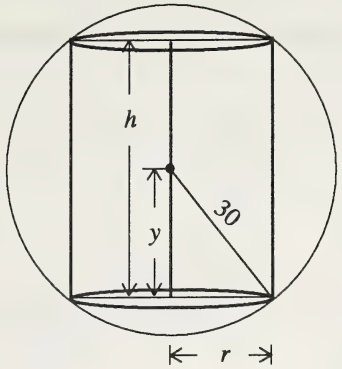
Maximum volume $= \frac{1}{3}\pi r^2 h$

$= \frac{1}{3}\pi(128) \times 16$

$= \frac{2048}{3}\pi$

The maximum volume is $\frac{2048}{3}\pi$ cm^3 or $682\frac{2}{3}\pi$ cm^3.

9.



$y^2 + r^2 = 30^2$

$r^2 = 30^2 - y^2$

Since $y = \frac{h}{2}$, $r^2 = 30^2 - \left(\frac{h}{2}\right)^2$

$= 900 - \frac{h^2}{4}$

$V = \pi r^2 h$

$= \pi\left(900 - \frac{h^2}{4}\right)h$

$= \pi\left(900h - \frac{h^3}{4}\right)$

$D_h V = \pi\left(900 - \frac{3h^2}{4}\right) = 0$

$900 - \frac{3h^2}{4} = 0$

$\frac{3h^2}{4} = 900$

$h^2 = 1200$

$h = 20\sqrt{3}$

At $h = 20\sqrt{3}$, $r = \sqrt{900 - \frac{(1200)}{4}}$

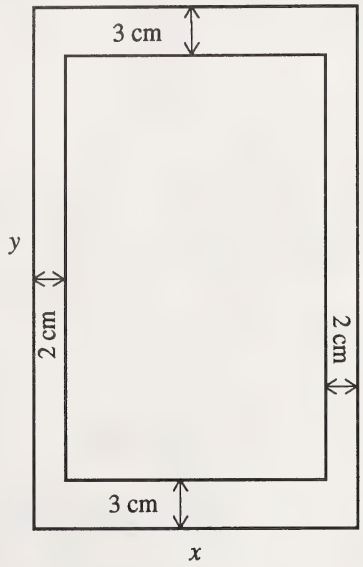
$= \sqrt{900 - 300}$

$= \sqrt{600}$

$= 10\sqrt{6}$

Therefore, the radius of the right circular cylinder is $10\sqrt{6}$ cm and the height is $20\sqrt{3}$ cm.

10.



$xy = 600$

$\therefore y = \frac{600}{x}$

Printed area $(A) = (y-6)(x-4)$

$$A = \left(\frac{600}{x} - 6\right)(x-4)$$

$$= 600 - 6x - \frac{2400}{x} + 24$$

$$= 624 - 6x - 2400x^{-1}$$

$$D_x A = -6 + \frac{2400}{x^2}$$

Let $D_x A = 0$.

$$-6 + \frac{2400}{x^2} = 0$$

$$6x^2 = 2400$$

$$x^2 = 400$$

$$x = 20$$

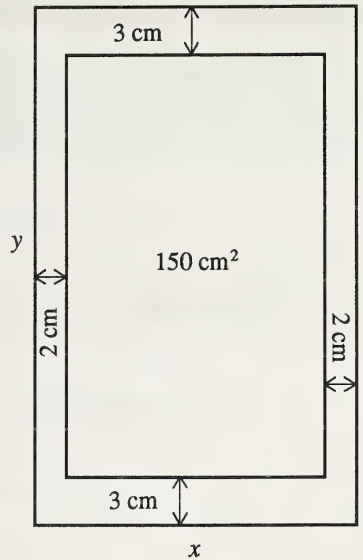
$$y = \frac{600}{x}$$

$$= \frac{600}{20}$$

$$= 30$$

The dimensions of the paper are 30 cm by 20 cm.

11.



Printed area $= 150$

$$= (x-4)(y-6)$$

Thus, $y = \frac{150}{x-4} + 6$.

Area of paper $(A) = xy$

$$= x\left(\frac{150}{x-4} + 6\right)$$

$$= \frac{150x}{x-4} + 6x$$

$$D_x A = \frac{(x-4)150 - 150x}{(x-4)^2} + 6$$

$$= \frac{-600}{(x-4)^2} + 6$$

Let $D_x A = 0$.

$$\frac{-600}{(x-4)^2} + 6 = 0$$

$$6 = \frac{600}{(x-4)^2}$$

$$6(x-4)^2 = 600$$

$$(x-4)^2 = 100$$

$$x - 4 = \pm 10$$

$x = 14, \ x = -6$ (Discard -6 because x cannot be negative.)

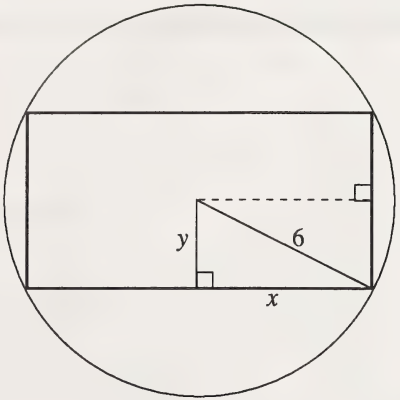
At $x = 14$, $y = \frac{150}{14-4} + 6$

$$= 15 + 6$$

$$= 21$$

The dimensions of the smallest page are 21 cm by 14 cm.

12.



$$x^2 + y^2 = 6^2$$
$$y^2 = 36 - x^2$$
$$y = \sqrt{36 - x^2}$$

Area of rectangle $(A) = (2x)(2y)$
$$= 4xy$$
$$= 4x\left(36 - x^2\right)^{\frac{1}{2}}$$

$$D_x A = 4x\left(\frac{1}{2}\right)\left(36 - x^2\right)^{-\frac{1}{2}}(-2x) + \left(36 - x^2\right)^{\frac{1}{2}}(4)$$
$$= -4x^2\left(36 - x^2\right)^{-\frac{1}{2}} + 4\left(36 - x^2\right)^{\frac{1}{2}}$$
$$= \frac{-4x^2 + 4\left(36 - x^2\right)}{\left(36 - x^2\right)^{\frac{1}{2}}}$$
$$= \frac{-8x^2 + 144}{\left(36 - x^2\right)^{\frac{1}{2}}}$$

Let $D_x A = 0$.

$$-8x^2 + 144 = 0$$
$$8x^2 = 144$$
$$x^2 = 18$$
$$x = 3\sqrt{2}$$

At $x = 3\sqrt{2}$, $y = \sqrt{36 - 18}$
$$= \sqrt{18}$$
$$= 3\sqrt{2}$$

The dimensions of the rectangle are $2x$ by $2y$; thus, its dimensions are $6\sqrt{2}$ cm by $6\sqrt{2}$ cm.

Activity 3

Determine other applications involving maxima and minima.

1. Let the two natural numbers be x and $(x+1)$.

$$S=\frac{4}{x}+(x+1)$$

$$D_x S=-\frac{4}{x^2}+1=0$$

$$\frac{4}{x^2}=1$$

$$4=x^2$$

$$x=2 \quad (x\neq -2 \text{ because } x \text{ is a natural number.})$$

When $x=2$, $x+1=3$.

Therefore, the two numbers are 2 and 3.

2. Let the two natural numbers be x and y.

$$x+y=4$$

$$y=4-x$$

$$S=x^3+y^3$$

$$=x^3+(4-x)^3$$

$$=x^3+64-48x+12x^2-x^3$$

$$=64-48x+12x^2$$

$$D_x S=-48+24x=0$$

$$24x=48$$

$$x=2$$

At $x=2$, $y=4-2$

$$=2$$

The two numbers are 2 and 2.

3. $C=250+\frac{h}{10}+\frac{2\,500\,000}{h}$

$$D_h C=\frac{1}{10}+(-1)2\,500\,000h^{-2}=0$$

$$\frac{2\,500\,000}{h^2}=\frac{1}{10}$$

$$h^2=25\,000\,000$$

$$h=5000$$

The height is 5000 m.

4. $$E=\frac{\left(m-\frac{1}{3}m^2\right)}{m+\frac{1}{3}}$$

$$D_m E=\frac{\left(m+\frac{1}{3}\right)D_m\left(m-\frac{1}{3}m^2\right)-\left(m-\frac{1}{3}m^2\right)D_m\left(m+\frac{1}{3}\right)}{\left(m+\frac{1}{3}\right)^2}$$

$$=\frac{\left(m+\frac{1}{3}\right)\left(1-\frac{2}{3}m\right)-\left(m-\frac{1}{3}m^2\right)}{\left(m+\frac{1}{3}\right)^2}$$

At a maximum, $D_m E=0$.

$$\therefore\frac{\left(m+\frac{1}{3}\right)\left(1-\frac{2}{3}m\right)-\left(m-\frac{1}{3}m^2\right)}{\left(m+\frac{1}{3}\right)^2}=0$$

$$\left(m+\frac{1}{3}\right)\left(1-\frac{2}{3}m\right)-\left(m-\frac{1}{3}m^2\right)=0$$

$$m-\frac{2}{3}m^2+\frac{1}{3}-\frac{2}{9}m-m+\frac{1}{3}m^2=0$$

$$-\frac{1}{3}m^2-\frac{2}{9}m+\frac{1}{3}=0$$

$$3m^2+2m-3=0$$

$$m=\frac{-2\pm\sqrt{2^2-4(3)(-3)}}{2\times3}$$

$$=\frac{-2\pm\sqrt{40}}{6}$$

$$=\frac{-2\pm2\sqrt{10}}{6}$$

$$=\frac{-1\pm\sqrt{10}}{3}$$

Since the tangent of the angle of pitch is always positive, $m=\frac{-1+\sqrt{10}}{3}$ is the value of m when E is a maximum.

5. $$I=\frac{E}{R+r}$$

$$P=I^2R$$

$$=\left(\frac{E}{R+r}\right)^2R$$

$$=E^2(R+r)^{-2}R$$

$$D_R P=E^2\left[R\,D_R(R+r)^{-2}+(R+r)^{-2}D_R R\right]$$

$$=E^2\left[R(-2)(R+r)^{-3}+(R+r)^{-2}\right]$$

$$=E^2\left[\frac{-2R}{(R+r)^3}+\frac{1}{(R+r)^2}\right]$$

$$=E^2\left[\frac{-2R+(R+r)}{(R+r)^3}\right]$$

At a maximum, $D_R P = 0$.

$$\therefore -2R + (R + r) = 0$$
$$-R + r = 0$$
$$R = r$$

If $r = 0.2\ \Omega$, then $R = 0.2\ \Omega$.

The external resistance is $0.2\ \Omega$ when the power is a maximum.

6. If $P = N + \dfrac{Nt}{t^2 + 64}$, then $P = 2000 + \dfrac{2000t}{t^2 + 64}$.

$$D_t P = 0 + \frac{\left(t^2 + 64\right) D_t (2000t) - 2000t\, D_t \left(t^2 + 64\right)}{\left(t^2 + 64\right)^2}$$

$$= \frac{\left(t^2 + 64\right) 2000 - 2000t(2t)}{\left(t^2 + 64\right)^2}$$

$$= \frac{2000t^2 + 128\,000 - 4000t^2}{\left(t^2 + 64\right)^2}$$

$$= \frac{-2000t^2 + 128\,000}{\left(t^2 + 64\right)^2}$$

At a maximum, $D_t P = 0$.

$$\therefore -2000t^2 + 128\,000 = 0$$
$$2000t^2 = 128\,000$$
$$t^2 = 64$$
$$t = 8$$

At $t = 8$, $P = 2000 + \dfrac{2000(8)}{8^2 + 64}$

$$= 2000 + \frac{16\,000}{128}$$
$$= 2000 + 125$$
$$= 2125$$

The maximum size of this population is 2125 when $t = 8$ h.

7. Assume that the rent is increased by $25x$, where $x \in N$. Rent per unit is equal to $460 + 25x$. The number of units occupied is equal to $150 - 4x$.

$$\text{Revenue} = (460 + 25x)(150 - 4x)$$
$$= 69\,000 + 3750x - 1840x - 100x^2$$
$$= -100x^2 + 1910x + 69\,000$$

$$\text{Cost} = (150 - 4x) \times 72.50$$
$$= 10\,875 - 290x$$

Profit (P) = revenue − cost

$$= -100x^2 + 1910x + 69\,000 - (-290x + 10\,875)$$

$$= -100x^2 + 1910x + 69\,000 + 290x - 10\,875$$

$$= -100x^2 + 2200x + 58\,125$$

At a maximum, $D_x P = 0$.

$$D_x P = -200x + 2200 = 0$$

$$200x = 2200$$

$$x = 11$$

Rent charged = \$460 + \$25(11)

= \$735

In order to realize the most profit, the rent charged should be \$735.

Extra Help

1. Step 1: Let l = length of the rectangle,
w = width of the rectangle, and
P = perimeter of the rectangle.
Area $(A) = l \times w$

Step 2: Perimeter = $2l + 2w = 100$ cm

$$l + w = 50 \text{ cm}$$

$$w = 50 - l$$

$$A = l \times w$$

$$= l(50 - l)$$

$$= 50l - l^2$$

Step 3: $D_l A = -2l + 50$

At a maximum, $D_l A = 0$.

$$\therefore -2l + 50 = 0$$

$$2l = 50$$

$$l = 25$$

$$w = 50 - l$$

$$= 50 - 25$$

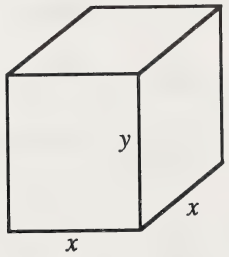
$$= 25$$

Step 4: $D_l^2 A = -2$, which is negative; thus, the area is a maximum.

The dimensions of the rectangle are 25 cm by 25 cm.

2. Step 1: Let the dimensions of the base be x by x.
Let y be the height of the box.

V (volume) $= x^2 y$



Step 2: Total area $= x^2 + 4xy = 48$

$$\therefore y = \frac{48 - x^2}{4x}$$

$$V = x^2 y$$
$$= x^2\left(\frac{48 - x^2}{4x}\right)$$
$$= \frac{48x - x^3}{4}$$

Step 3: At a maximum, $D_x V = 0$.

$$\therefore D_x V = 12 - \frac{3}{4}x^2 = 0$$
$$\frac{3}{4}x^2 = 12$$
$$x^2 = 16$$
$$x = 4$$

$$\text{At } x = 4,\ y = \frac{48 - 4^2}{4(4)}$$
$$= \frac{48 - 16}{16}$$
$$= 2$$

Step 4: $D_x^2 V = -\frac{3}{2}x$

At $x = 4$, $D_x^2 V = -6$.

Since the second derivative is negative, it represents a maximum volume.

The dimensions of the box are 4 cm × 4 cm × 2 cm and the maximum volume is 32 cm^3.

Extensions

$$y = x^4 + 4x^3 - 8x^2 - 48x + 4$$

$$D_x y = 4x^3 + 12x^2 - 16x - 48 = 0$$
$$x^3 + 3x^2 - 4x - 12 = 0$$

Factoring gives you the following:

$$x^2(x+3) - 4(x+3) = 0$$
$$\left(x^2 - 4\right)(x+3) = 0$$
$$(x+3)(x+2)(x-2) = 0$$

$$x + 3 = 0,\quad x + 2 = 0,\quad x - 2 = 0$$
$$x = -3 \qquad x = -2 \qquad x = 2$$

$$D_x^2 y = 12x^2 + 24x - 16$$

$$\text{At } x = -3,\ y = (-3)^4 + 4(-3)^3 - 8(-3)^2 - 48(-3) + 4$$
$$= 81 - 108 - 72 + 144 + 4$$
$$= 49$$

$$D_x^2 y = 12(-3)^2 + 24(-3) - 16$$
$$= 108 - 72 - 16$$
$$= 20 \quad \text{(positive)}$$

Thus, 49 is a minimum when $x = -3$.

$$\text{At } x = -2,\ y = (-2)^4 + 4(-2)^3 - 8(-2)^2 - 48(-2) + 4$$
$$= 16 - 32 - 32 + 96 + 4$$
$$= 52$$

$$D_x^2 y = 12(-2)^2 + 24(-2) - 16$$
$$= 48 - 48 - 16$$
$$= -16 \quad \text{(negative)}$$

Thus, 52 is a maximum.

At $x = 2$, $y = (2)^4 + 4(2)^3 - 8(2)^2 - 48(2) + 4$
$$= 16 + 32 - 32 - 96 + 4$$
$$= -76$$

$$D_x^2 y = 12(2)^2 + 24(2) - 16$$
$$= 48 + 48 - 16$$
$$= 80 \quad \text{(positive)}$$

Thus, -76 is a minimum.

Exploring Topic 2

Activity 1

Define velocity and solve related problems.

1.

a. 150 km

b. Average velocity $= \dfrac{150}{1.5 + 2.5 + 3}$
$$= \frac{150}{7}$$
$$= 21\frac{3}{7}$$

The average velocity is $21\frac{3}{7}$ km/h.

c. Average speed $= \dfrac{150 + 250 + 250}{1.5 + 2.5 + 3}$
$$= \frac{650}{7}$$
$$\doteq 92.9$$

The average speed is about 92.9 km/h.

2. a. Displacement $= (90 \times 3 + 100 \times 4)$
$$= 270 + 400$$
$$= 670$$

The displacement is 670 km.

b. Average velocity $= \frac{670}{7}$

$= 95\frac{5}{7}$

The average velocity is $95\frac{5}{7}$ km/h.

3. a. $h = -5t^2 + 45t + 25$

$v = \frac{dh}{dt} = -10t + 45$

b. At $t = 1$, $v = -10(1) + 45$

$= 35$

The velocity at $t = 1$ is 35 m/s.

c. When $v = 0$, the object reaches its maximum displacement.

$-10t + 45 = 0$

$10t = 45$

$t = 4.5$

When the object reaches its maximum displacement, $t = 4.5$ s.

d. At $t = 4.5$, $h = -5(4.5)^2 + 45(4.5) + 25$

$= -101.25 + 202.5 + 25$

$= 126.25$

The maximum displacement is 126.25 m.

e. At $t = 2$, $h = -5(2)^2 + 45(2) + 25$

$= -20 + 90 + 25$

$= 95$

At $t = 0$, $h = -5(0)^2 + 45(0) + 25$

$= 0 + 0 + 25$

$= 25$

Average velocity $= \frac{95 - 25}{2 - 0}$

$= \frac{70}{2}$

$= 35$

The average velocity from $t = 0$ to $t = 2$ is 35 m/s.

4. a. $s = t^2 - 10t + 25$

At $t = 1$, $s = 1^2 - 10(1) + 25$

$= 1 - 10 + 25$

$= 16$

At $t = 4$, $s = 4^2 - 10(4) + 25$

$= 16 - 40 + 25$

$= 1$

Average velocity $= \frac{1 - 16}{4 - 1}$

$= \frac{-15}{3}$

$= -5$

The average velocity from $t = 1$ to $t = 4$ is 5 m/s.

b. $\frac{ds}{dt} = 2t - 10$

c. At $t = 3$, $\frac{ds}{dt} = 2(3) - 10$

$= -4$

The velocity at $t = 3$ is -4 m/s.

d. At a minimum, $\frac{ds}{dt} = 0$.

$\therefore 2t - 10 = 0$

$t = 5$

e. At $t = 5$, $s = 5^2 - 10(5) + 25$

$= 0$

The minimum displacement is 0 m.

Activity 2

Define acceleration and solve related problems.

1. $a = \frac{dv}{dt} = 2 - 6t$

At $t = 4$, $a = 2 - 6(4)$

$= 2 - 24$

$= -22$

2. $a = \frac{dv}{dt} = 2(t + 3)$

At $t = 3$, $a = 2(3 + 3)$

$= 12$

3. $a = \frac{dv}{dt} = 15t^2 - 2$

At $t = 2$, $a = 15(2)^2 - 2$

$= 60 - 2$

$= 58$

4. $a = \frac{dv}{dt} = \frac{\left(3 + t^2\right) D_t(10t) - (10t) D_t\left(3 + t^2\right)}{\left(3 + t^2\right)^2}$

$= \frac{\left(3 + t^2\right)(10) - (10t)(2t)}{\left(3 + t^2\right)^2}$

$= \frac{30 + 10t^2 - 20t^2}{\left(3 + t^2\right)^2}$

$= \frac{-10t^2 + 30}{\left(3 + t^2\right)^2}$

At $t=1$, $a=\dfrac{-10(1)^2+30}{\left(3+1^2\right)^2}$

$=\dfrac{20}{16}$

$=1\dfrac{1}{4}$

5. $s=t^2-5t+4$

a. $v=D_t s=2t-5$

$a=D_t v=2$

b. $2t-5=0$

$t=\dfrac{5}{2}$ or $2\dfrac{1}{2}$

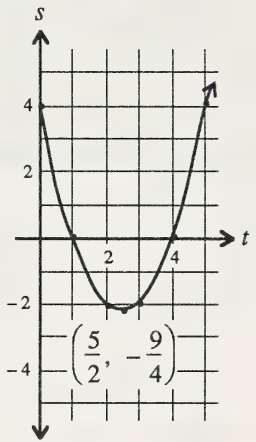
At $t=\dfrac{5}{2}$, $s=\left(\dfrac{5}{2}\right)^2-5\left(\dfrac{5}{2}\right)+4$

$=-2\dfrac{1}{4}$

Therefore, the turning point is $\left(\frac{5}{2},\ -2\frac{1}{4}\right)$.

c.

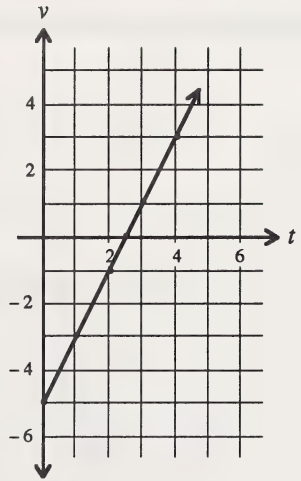
t	0	1	2	$2\frac{1}{2}$	3	4
s	4	0	-2	$-2\frac{1}{4}$	-2	0



d. At $t=1$, the direction of the motion is negative.
At $t=\frac{5}{2}$, the object stops.
At $t=3$, the direction of the motion is positive.

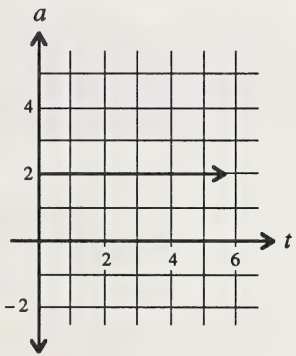
e.

t	0	1	2	$\frac{5}{2}$	3	4
v	-5	-3	-1	0	1	3



The velocity is increasing at a constant rate.

f. $a = \frac{dv}{dt} = 2$



The acceleration is a constant 2.

6. a. $s = 2t^3 - 15t^2 + 24t + 8,\ t \geq 0$

$$v = \frac{ds}{dt} = 6t^2 - 30t + 24$$

$$a = \frac{dv}{dt} = 12t - 30$$

b. $6t^2 - 30t + 24 = 0$

$$t^2 - 5t + 4 = 0$$

$$(t-4)(t-1) = 0$$

$$t = 4,\ t = 1$$

At $t = 4$, $s = 2(4)^3 - 15(4)^2 + 24(4) + 8$

$$= 128 - 240 + 96 + 8$$

$$= -8$$

At $t = 1$, $s = 2(1)^3 - 15(1)^2 + 24(1) + 8$

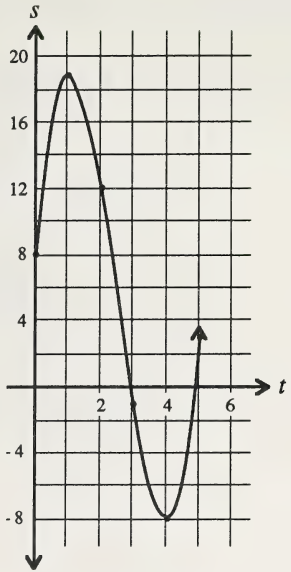
$$= 2 - 15 + 24 + 8$$

$$= 19$$

The two turning points are (4, – 8) and (1, 19).

c.

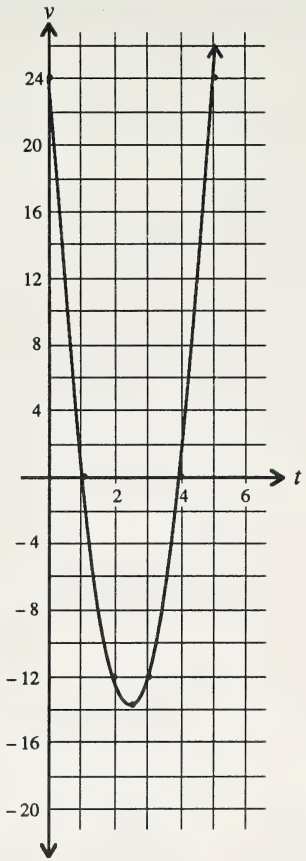
t	0	1	2	3	4	5
s	8	19	12	– 1	– 8	3



d. At $t = 0$, the direction of the motion is positive.
At $t = 1$, the object stops.
At $t = 3$, the direction of the motion is negative.
At $t = 4$, the object stops.
At $t = 5$, the direction of the motion is positive.

e. $v = 6t^2 - 30t + 24$

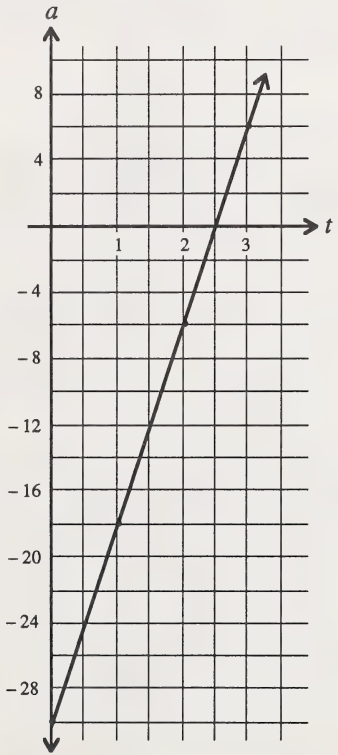
t	0	1	2	3	4	5	$2\frac{1}{2}$
v	24	0	-12	-12	0	24	$-13\frac{1}{2}$



The velocity is decreasing from $t = 0$ to $t = 2\frac{1}{2}$. At $t = 2\frac{1}{2}$, the velocity is a minimum. After that, the velocity is increasing.

f. $a = 12t - 30$

t	0	1	2	3
a	-30	-18	-6	6



The acceleration is increasing at a constant rate.

7. $v = \frac{ds}{dt} = 9t^2 - 6t + 1$

$a = \frac{dv}{dt} = 18t - 6$

When $v = 4$ m/s, $9t^2 - 6t + 1 = 4$

$$9t^2 - 6t - 3 = 0$$

$$3\left(3t^2 - 2t - 1\right) = 0$$

$$3(t-1)(3t+1) = 0$$

$t - 1 = 0, \quad 3t + 1 = 0$

$t = 1 \qquad t = -\frac{1}{3}$ (Discard; $t \geq 0$.)

At $t = 1$, $a = 18(1) - 6$

$a = 12$

When the velocity is 4 m/s, the acceleration is 12 m/s^2.

8. $s=\dfrac{t^2}{3+t}$

$$v=\frac{ds}{dt}=\frac{(3+t)D_t\left(t^2\right)-t^2D_t(3+t)}{(3+t)^2}$$

$$=\frac{(3+t)(2t)-t^2(1)}{(3+t)^2}$$

$$=\frac{6t+1t^2}{(3+t)^2}$$

If $v=\dfrac{3}{4}$ m / s, then $\dfrac{6t+t^2}{(3+t)^2}=\dfrac{3}{4}$.

$$\left(6t+t^2\right)(4)=3(3+t)^2$$

$$24t+4t^2=3\left(9+6t+t^2\right)$$

$$24t+4t^2=27+18t+3t^2$$

$$t^2+6t-27=0$$

$$(t-3)(t+9)=0$$

$t-3=0,\ \ t+9=0$

$t=3 \qquad t=-9$ (Discard; $t\geq 0$.)

$$a=\frac{dv}{dt}=\frac{(3+t)^2D_t\left(6t+t^2\right)-\left(6t+t^2\right)D_t(3+t)^2}{(3+t)^4}$$

$$=\frac{(3+t)^2(6+2t)-\left(6t+t^2\right)(2)(3+t)}{(3+t)^4}$$

$$=\frac{\left(9+6t+t^2\right)(6+2t)-\left(6t+t^2\right)(6+2t)}{(3+t)^4}$$

$$=\frac{(6+2t)\left(9+6t+t^2-6t-t^2\right)}{(3+t)^4}$$

$$=\frac{9(6+2t)}{(3+t)^4}$$

$$=\frac{18(3+t)}{(3+t)^4}$$

$$=\frac{18}{(3+t)^3}$$

At $t=3$, $a=\dfrac{18}{(3+3)^3}$

$$=\frac{18}{6^3}$$

$$=\frac{1}{12}$$

When the velocity is $\frac{3}{4}$ m / s, the acceleration is $\frac{1}{12}$ m / s^2.

Activity 3

Use implicit differentiation to solve motion problems.

1. $D_t\left(3v^2\right) = D_t(18s) + D_t(300)$

$$6v\frac{dv}{dt} = 18\frac{ds}{dt} + 0$$

Since $\frac{dv}{dt} = a$ and $\frac{ds}{dt} = v$,

$$6va = 18v$$

$$a = \frac{18v}{6v}$$

$$= 3.$$

2. $D_t(E) = D_t\left(\frac{1}{2}mv^2\right) + D_t(mps)$

$$0 = \frac{1}{2}m(2v)D_t v + mp\,D_t s$$

$$= mva + mpv$$

$$mva = -mpv$$

$$a = -\frac{mpv}{mv}$$

$$= -p$$

3. $a = \frac{dv}{dt} = D_t\left(\frac{6000s}{6+s}\right)$

$$= \frac{(6+s)D_t(6000s) - (6000s)D_t(6+s)}{(6+s)^2}$$

$$= \frac{(6+s)(6000)\frac{ds}{dt} - (6000s)\frac{ds}{dt}}{(6+s)^2}$$

$$= \frac{36\,000v + 6000vs - 6000vs}{(6+s)^2}$$

$$= \frac{36\,000v}{(6+s)^2}$$

$$= \frac{36\,000\left(\frac{6000s}{6+s}\right)}{(6+s)^2}$$

$$= \frac{216\,000\,000s}{(6+s)^3}$$

4. $D_t\left(\frac{1}{v}\right) + D_t\left(\frac{1}{3}\right) = D_t(4t)$

$$-\frac{1}{v^2}\frac{dv}{dt} + 0 = 4$$

$$-\frac{a}{v^2} = 4$$

$$-a = 4v^2$$

$$a = -4v^2$$

5. $D_x\sqrt{(y^2-2)} = D_x(xy)$

$$\frac{1}{2}(y^2-2)^{-\frac{1}{2}} D_x(y^2-2) = x\frac{dy}{dx} + y$$

$$\frac{2y D_x y}{2(y^2-2)^{\frac{1}{2}}} = x D_x y + y$$

$$2y D_x y = 2x(y^2-2)^{\frac{1}{2}} D_x y + 2y(y^2-2)^{\frac{1}{2}}$$

$$\left[2y - 2x(y^2-2)^{\frac{1}{2}}\right] D_x y = 2y(y^2-2)^{\frac{1}{2}}$$

$$D_x y = \frac{2y(y^2-2)^{\frac{1}{2}}}{2y - 2x(y^2-2)^{\frac{1}{2}}}$$

$$= \frac{y(y^2-2)^{\frac{1}{2}}}{y - x(y^2-2)^{\frac{1}{2}}}$$

6. $D_x(x^3 y) - D_x(4y^2) = D_x(3)$

$$x^3 D_x y + y(3x^2) - 8y D_x y = 0$$

$$(x^3 - 8y) D_x y = -3x^2 y$$

$$D_x y = \frac{-3x^2 y}{x^3 - 8y} \text{ or } \frac{3x^2 y}{8y - x^3}$$

Extra Help

1. a. $v = \frac{ds}{dt} = 6t^2 - 42t + 60$

At $t = 2$, $v = 6(2)^2 - 42(2) + 60$

$= 24 - 84 + 60$

$= 0$

Remember that s is a maximum or minimum when $v = 0$. When $a = 0$, v is a maximum or minimum.

b. $a = \frac{dv}{dt} = 12t - 42$

At $t = 3$, $a = 12(3) - 42$

$= 36 - 42$

$= -6$

c. At $t = 1$, $s = 2(1)^3 - 21(1)^2 + 60(1) + 80$

$= 2 - 21 + 60 + 80$

$= 121$

At $t = 3$, $s = 2(3)^3 - 21(3)^2 + 60(3) + 80$

$= 54 - 189 + 180 + 80$

$= 125$

v (average velocity) $= \frac{125 - 121}{3 - 1}$

$= \frac{4}{2}$

$= 2$

The average velocity is 2 m/s.

d. When $a = 0$, v is a maximum or minimum.

$$12t - 42 = 0$$
$$12t = 42$$
$$t = \frac{42}{12}$$
$$= 3.5$$

At $t = 3.5$, $v = 6(3.5)^2 - 42(3.5) + 60$
$$= 73.5 - 147 + 60$$
$$= -13.5$$

If the coefficient of the second-degree term in a quadratic function is positive, then the function has a minimum value. If the coefficient of the second-degree term in a quadratic function is negative, then the function has a maximum value.

The minimum velocity is – 13.5 m/s.

$v = 6t^2 - 42t + 60$ (The coefficient of t^2 is positive; thus, it is a minimum.)

2. $x^2 + 2xy + 4y^2 = 3$

$$D_x\left(x^2\right) + D_x(2xy) + D_x\left(4y^2\right) = D_x(3)$$
$$2x + 2x\,D_x y + 2y + 8y\,D_x y = 0$$
$$(2x + 8y)D_x y = -2x - 2y$$
$$D_x y = \frac{-2x - 2y}{2x + 8y}$$
$$= \frac{-x - y}{x + 4y}$$

3. $D_t\left(3v^2\right) = D_t(5s) + D_t(1000)$

$$6v\,D_t v = 5\,D_t s + 0$$
$$6va = 5v$$
$$a = \frac{5v}{6v}$$
$$= \frac{5}{6}$$

Extensions

1. $D_t\left(s^2\right) + 25\,D_t\left(v^2\right) = 0$

$$2s\,D_t s + 50v\,D_t v = 0$$

$D_t s = v$ and $D_t v = a$

$$2sv + 50va = 0$$
$$s = \frac{-50va}{2v}$$
$$s = -25a$$

2. $D_t v = \dfrac{(3s^2-2)D_t(1-3s^2)-(1-3s^2)D_t(3s^2-2)}{(3s^2-2)^2}$

$D_t v = \dfrac{(3s^2-2)(-6s\,D_t s)-(1-3s^2)6s\,D_t s}{(3s^2-2)^2}$

Since $D_t s = v$ and $D_t v = a$,

$a = \dfrac{(3s^2-2)(-6sv)-(1-3s^2)(6sv)}{(3s^2-2)^2}.$

By factoring you get the following:

$a = \dfrac{-6sv(3s^2-2-3s^2+1)}{(3s^2-2)^2}$

$= \dfrac{6sv}{(3s^2-2)^2}$

Since $v = \dfrac{1-3s^2}{3s^2-2}$ (from the original equation),

$a = \dfrac{6s(1-3s^2)}{(3s^2-2)^3}.$

Exploring Topic 3

Activity 1

Solve applied problems involving rates of change.

1. $A = \pi r^2$

$\dfrac{dA}{dt} = 2\pi r \dfrac{dr}{dt}$

You are given $\frac{dA}{dt} = \frac{\pi}{2}$ cm^2 / min and $A = 9\pi$ cm^2.

$\dfrac{dr}{dt} = \dfrac{1}{2\pi r}\dfrac{dA}{dt}$

$= \dfrac{1}{2\pi r} \times \dfrac{\pi}{2}$

$= \dfrac{1}{4r}$

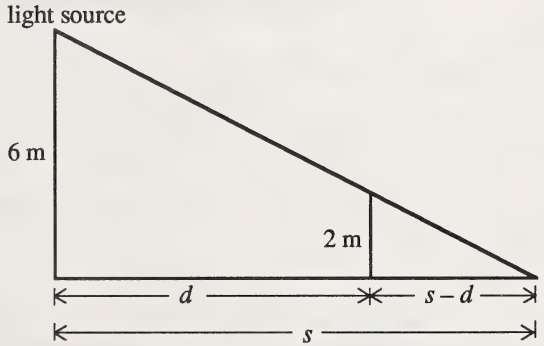
When the area of the circle is 9π cm^2, the radius of the circle is as follows:

$$r = \sqrt{\frac{9\pi}{\pi}}$$
$$= \sqrt{9}$$
$$= 3$$

$$\therefore \frac{dr}{dt} = \frac{1}{4r}$$
$$= \frac{1}{4\times 3}$$
$$= \frac{1}{12}$$

The radius of the circle is increasing at the rate of $\frac{1}{12}$ cm/min.

2.



$D_t d = 1.5$ m/s (given)

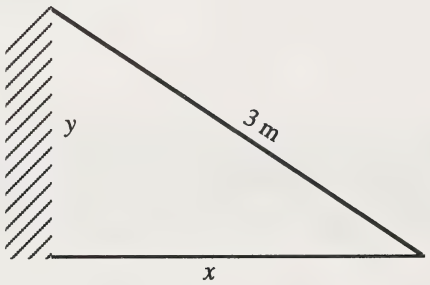
In the preceding diagram the two triangles are similar and their corresponding sides are proportional.

$$\frac{2\text{ m}}{6\text{ m}} = \frac{s-d}{s}$$

$$2s = 6s - 6d$$
$$D_t(2s) = D_t(6s) - D_t(6d)$$
$$2D_t s = 6D_t s - 6D_t d$$
$$4D_t s = 6D_t d$$
$$4D_t s = 6(1.5)$$
$$4D_t s = 9$$
$$D_t s = \frac{9}{4} \text{ or } 2.25$$

The end of the shadow of the post is moving away at 2.25 m/s.

3.



$\frac{dx}{dt} = 0.5$ m/s (given)

$$x^2 + y^2 = 3^2$$

$$2x\,D_t x + 2y\,D_t y = 0$$

$$2x\left(\frac{1}{2}\right) + 2y\,D_t y = 0$$

$$D_t y = \frac{-x}{2y}$$

When $y = 1$, $x^2 + 1 = 9$

$$x^2 = 8$$

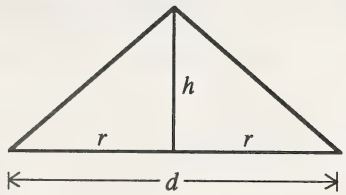
$$x = 2\sqrt{2}$$

$$D_t y = \frac{-2\sqrt{2}}{2(1)}$$

$$= -\sqrt{2}$$

The top of the post is moving down the wall at $\sqrt{2}$ m/s.

4.



$$\frac{dV}{dt} = 8 \text{ m}^3/\text{min} \text{ (given)}$$

$$d = 2r = 3h$$

$$\therefore r = \frac{3}{2}h$$

$$V = \frac{1}{3}\pi r^2 h$$

$$= \frac{1}{3}\pi\left(\frac{3}{2}h\right)^2 h$$

$$= \frac{3}{4}\pi h^3$$

$$D_t V = \frac{3}{4}\pi\left(3h^2\right) D_t h$$

$$8 = \frac{3}{4}\pi\left(3h^2\right) D_t h$$

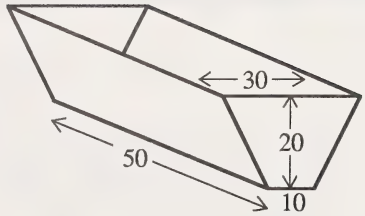
$$D_t h = \frac{32}{9\pi h^2}$$

When $h = 4$, $D_t h = \frac{32}{9\pi(4)^2}$

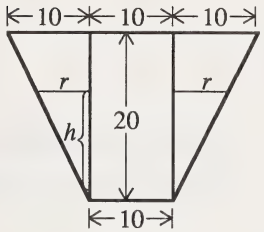
$$= \frac{2}{9\pi}$$

The height is increasing at $\frac{2}{9\pi}$ m/min.

5.



Area of a trapezoid $= \frac{1}{2}(a+b)h$



$\frac{r}{h} = \frac{10}{20}$

$\therefore r = \frac{1}{2}h$

The following is the area of the trapezoid when the height is h cm.

$$A = \frac{1}{2}[10+(2r+10)]h$$

$$= \frac{1}{2}(20+2r)h$$

$$= (10+r)h$$

$$= \left(10+\frac{h}{2}\right)h \qquad \left(\text{Substitute } r = \tfrac{1}{2}h.\right)$$

$$= 10h + \frac{h^2}{2}$$

Volume = area of trapezoid × length

$$V = \left(10h + \frac{h^2}{2}\right) \times 50$$

$$= 500h + 25h^2$$

$$D_t V = 500\, D_t h + 50h\, D_t h$$

$$= (500+50h) D_t h$$

Since $D_t V = 600 \text{ cm}^3/\text{s}$,

$$600 = (500+50h) D_t h$$

$$D_t h = \frac{600}{500+50h}$$

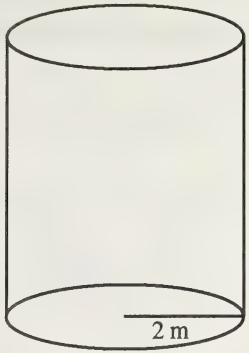
When $h = 10$, $D_t h = \frac{600}{500+500}$

$$= \frac{600}{1000}$$

$$= 0.6$$

The water is rising at 0.6 cm/s.

6.



$V = \pi r^2 h$

$= \pi(2)^2 h$

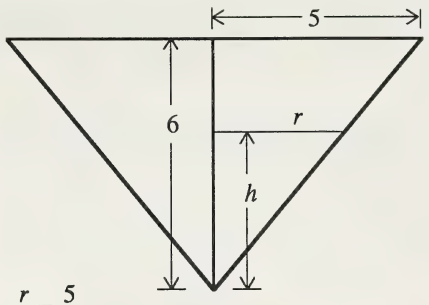
$= 4\pi h$

$D_t V = 4\pi D_t h \quad \left(D_t V = 3 \text{ m}^3 / \text{min}\right)$

$D_t h = \frac{3}{4\pi} \text{ m / min}$

The water level is rising at a constant rate of $\frac{3}{4\pi}$ m/min. The depth of 4 m is not needed because it does not affect $\frac{dh}{dt}$.

7.



$\frac{r}{h} = \frac{5}{6}$

$\therefore r = \frac{5}{6}h$

$V = \frac{1}{3}\pi r^2 h$

$= \frac{1}{3}\pi\left(\frac{5}{6}h\right)^2 h$

$= \frac{1}{3} \times \frac{25}{36}\pi h^3$

$= \frac{25}{108}\pi h^3$

$D_t V = \frac{25}{108}\pi\left(3h^2\right) D_t h \quad \left(D_t V = 3\pi \text{ m}^3 / \text{min}\right)$

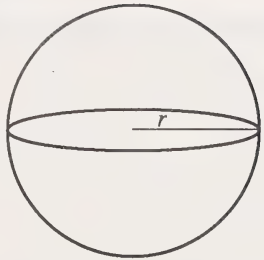
When $h = 3$, $3\pi = \frac{75}{108}\pi(3)^2 D_t h$

$D_t h = \frac{3\pi(12)}{75\pi}$

$= \frac{12}{25}$

The water level is rising $\frac{12}{25}$ m/min.

8.



$V = \frac{4}{3}\pi r^3$

$D_t V = \frac{4}{3}\pi\left(3r^2\right) D_t r$

Since $D_t V = 100 \text{ cm}^3/\text{s}$ and $r = 5$ cm,

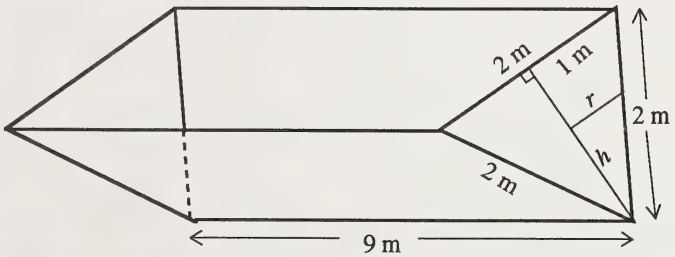
$100 = 4\pi(5)^2 D_t r$

$D_t r = \frac{100}{100\pi}$

$= \frac{1}{\pi}$

The radius is increasing at $\frac{1}{\pi}$ cm/s.

9.



Height of the triangle $= \sqrt{2^2 - 1^2}$ (Pythagorean property)

$= \sqrt{3}$

$\frac{\sqrt{3}}{1} = \frac{h}{r}$

$\therefore r = \frac{1}{\sqrt{3}}h$

$V =$ area of $\Delta \times$ length

$V = \frac{1}{2}(2r)h \times 9$

$= 9rh$

$= \left(\frac{9}{\sqrt{3}}h\right)h$

$= \frac{9}{\sqrt{3}}h^2$

$D_t V = \frac{9}{\sqrt{3}}(2h) D_t h$

Since $D_t V = 3 \text{ m}^3/\text{min}$ and $h = 0.5$ m,

$3 = \frac{9}{\sqrt{3}}\left(2 \times \frac{1}{2}\right) D_t h$

$D_t h = \frac{3\sqrt{3}}{9}$

$= \frac{\sqrt{3}}{3}$

The water level is rising at $\frac{\sqrt{3}}{3}$ m/min.

10. $D_t y = 4x\, D_t x$

Since $D_t x = 4$ and $x = 3$,

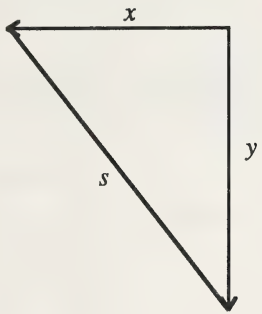
$D_t y = 4(3)(4)$

$= 48$

At $x = 3$, y is changing at 48 units/s.

Extra Help

1.



After 2 h, $x = 2 \times 50 = 100$ and $y = 2 \times 80 = 160$.

$s^2 = x^2 + y^2$

$2s\, D_t s = 2x\, D_t x + 2y\, D_t y$

Since $D_t x = 50$ km / h,

$D_t y = 80$ km / h.

If $x = 100$, and $y = 160$, then $s \doteq 188.68$.

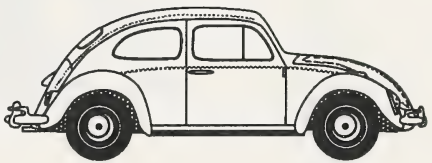
$s\, D_t s = x\, D_t x + y\, D_t y$

$188.68\, D_t s \doteq 100(50) + 160(80)$

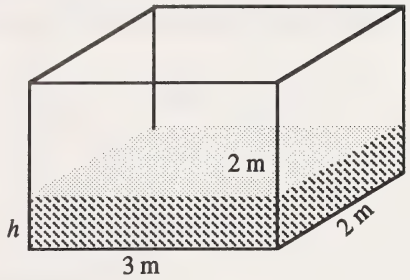
$D_t s \doteq \dfrac{17\,800}{188.68}$

$\doteq 94.34$

They are separating at approximately 94.34 km/h after 2 h.



2.



When the water is h m deep,

$$V = l \times w \times h$$
$$= 3 \times 2 \times h$$
$$= 6h$$

$$D_t V = 6\, D_t h$$

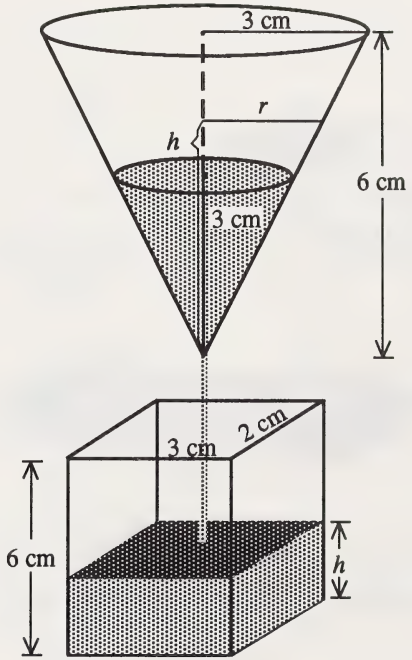
Since $D_t V = 1.5 \text{ m}^3 / \text{min}$, $1.5 = 6\, D_t h$

$$D_t h = \frac{1.5}{6}$$
$$= 0.25$$

The water is rising at 0.25 m/min.

Extensions

1.



Volume of a cone $(V) = \frac{1}{3}\pi r^2 h$

$$\frac{r}{h} = \frac{3}{6}$$
$$r = \frac{1}{2}h$$

$$V = \frac{1}{3}\pi r^2 h$$

$$= \frac{1}{3}\pi\left(\frac{h}{2}\right)^2 h$$

$$= \frac{\pi}{12}h^3$$

$$\frac{dV}{dt} = \frac{\pi}{12}\left(3h^2\right)\frac{dh}{dt}$$

At $h = 3$ cm, $\frac{dh}{dt} = \frac{1}{2}$ cm/s

$$\frac{dV}{dt} = \frac{9\pi}{4} \times \left(\frac{1}{2}\right)$$

$$= \frac{9\pi}{8} \text{ cm}^3/\text{s}$$ (the rate of flow out of the filter)

For the rectangular container, $V = l \times w \times h$

$= 3 \text{ cm} \times 2 \text{ cm} \times h.$

$$D_t V = 6\, D_t h$$

$$\frac{9\pi}{8} = 6\, D_t h$$

$$D_t h = \frac{9\pi}{8 \times 6}$$

$$= \frac{3\pi}{16}$$

The water level in the rectangular container is rising $\frac{3\pi}{16}$ cm/s.

Mathematics 31

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